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LYAPUNOV STABILITY FOR PARTIAL DIFFERENTIAL EQUATIONS

Part I - Lyapunov Stability Theory and the Stability of Solutions to Partial Differential Equations

by Gabe R. Buis

Part II - Contraction Groups and Equivalent Norms

by William G. Vogt, Martin M. Eisen, and Gabe R. Buis

Prepared by
UNIVERSITY OF PITTSBURGH
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PREFACE

This report is divided into two parts. Part I, "Lyapunov Stability Theory and the Stability of Solutions to Partial Differential Equations" is a slight revision of the Doctoral Dissertation of Dr. Gabe R. Buis which was submitted to and approved by the Graduate Faculty of the School of Engineering, University of Pittsburgh, September, 1967. This research was directed by his Major Advisor, Dr. William G. Vogt, Associate Professor of Electrical Engineering and Principal Investigator of the grant. Part II, "Contraction Groups and Equivalent Norms," by William G. Vogt, Martin M. Eisen and Gabe R. Buis presents further extensions of some of the research reported in Part I.

This research was supported under the National Aeronautics and Space Administration Grant No. NGR 39-011-039 with the University of Pittsburgh.

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PART II

CONTRACTION GROUPS AND EQUIVALENT NORMS

PART I

LYAPUNOV STABILITY THEORY AND THE STABILITY OF SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS

by

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ABSTRACT

Lyapunov stability theory is generalized to semi-groups and groups of linear operators in Hilbert spaces. The existence of a Lyapunov functional is sufficient for the asymptotic stability of semi-groups and necessary and sufficient for the asymptotic stability of groups. This theory is applied to a class of partial differential equations, yielding stability conditions which are mathematically rigorous.

SUMMARY

Lyapunov's stability theory has become very important in the stability analysis of solutions to ordinary differential equations. Its extension to partial differential equations has been restricted to a few scattered applications which generally lack mathematical rigor. The complex mathematical nature of partial differential equations makes extension of the stability theory to partial differential equations very difficult. The objective of this dissertation is to develop a mathematically rigorous Lyapunov stability theory for a class of partial differential equations.

The approach taken to this problem is that of generalizing Lyapunov's Direct Method for ordinary differential equations to a class of operator differential equations. This generalization is based on the fundamental solution structure of groups and semi-groups, which is not restricted to ordinary differential equations.

This leads to the formulation of sufficient conditions on an operator to generate stable or asymptotically stable semi-groups and necessary and sufficient conditions on an operator to generate asymptotically stable groups.

The semi-group and group structures enable one to associate with the infinitesimal generator an operator differential equation. The derived stability conditions extend then to the stability of the null solution of this differential equation. The generators of these semi-groups and groups constitute a large class of bounded and unbounded operators.

The second stage is to formulate partial differential equations in the framework of the operator differential equations. Starting with a formal partial differential operator, distributions are introduced to define the

extended operator in a complete space. The next step is to associate boundary conditions with the formal partial differential operator. This is limited to the so-called Dirichlet boundary conditions, which are important for many stability investigations.

For the class of strongly elliptic partial differential operators the domain and range are formulated in terms of Sobolev spaces. Subsequently, the developed stability theory is applied to a class of evolution equations with strongly elliptic partial differential operators and Dirichlet boundary conditions, giving sufficient conditions for asymptotic stability of the null solution.

A similar formulation is given for a class of wave equations. However, for this case the necessary and sufficient conditions for asymptotic stability of the null solution are established by developing a suitable Lyapunov Functional. At the same time this proves that the solutions possess the group property. Various applications are given. The stability analysis is further extended to some nonlinear partial differential equations. With the exception of these last results the emphasis has been on a mathematically rigorous formulation of the stability problem.

I. INTRODUCTION

In recent years Lyapunov stability theory has become an important tool in the stability analysis of solutions to linear and nonlinear ordinary differential equations. The original work of Lyapunov[1]* has generated many contributions to the stability theory of solutions to ordinary differential equations and has provided many applications. Two of the more significant ones are the paper by Kalman and Bertram[2] and the book by LaSalle and Lefschetz[3]. Where these contributions involve Lyapunov's Direct Method, the central problem becomes the construction of a Lyapunov function. For nonlinear systems this is generally very difficult.

Although the development of Lyapunov stability theory and applications to solutions of ordinary differential equations has progressed rapidly, its application to solutions of partial differential equations has remained limited. This is the case despite the fact that many physical systems must be represented by partial differential equations. However, the difficulties encountered in applying Lyapunov stability theory to partial differential equations parallel those in establishing the existence and uniqueness properties of solutions to partial differential equations.

Yet stability remains one of the most important properties of distributed parameter systems. Many of the stability results for partial differential equations are obtained by using methods of approximation. These methods might not give sufficient conditions for stability except in the case of infinitesimally small perturbations. Stability can be defined in many different ways. In reference to stability theory it will be interpreted here as stability in

*Parenthetical references placed superior to the line of text refer to the bibliography.

the sense of Lyapunov: a system is said to be stable if for sufficiently small initial perturbations the solutions remain close to the original solution for all future time. The advantages of Lyapunov's Direct Method over approximate methods are many. Most important, Lyapunov's Direct Method employs the system equations directly without resorting to approximations. The method also allows mathematical rigor and in principle the introduction of nonlinearities.

On the other hand, there is the importance of partial differential equations in the fields of reactor physics, hydrodynamics, magnetohydrodynamics, control processes, etc. These facts certainly motivate an investigation of possible ways to extend Lyapunov stability theory for the stability analysis of solutions to partial differential equations. The following section gives a survey of the significant results obtained so far. This survey shows how limited these results are. The survey also points out the lack of mathematical rigor in many of the applications. This section is followed by a short outline of general problem areas in the stability study of solutions to partial differential equations.

A. Review of the Literature

Many stability results for partial differential equations are derived by using approximate methods. The basis for these approximate methods is the reduction of the partial differential equations to a system of ordinary differential equations. This can be done by either approximating the model by one having a finite number of degrees of freedom via spatial discretization or by assuming a harmonic time dependence. The first case allows the application of the well-known techniques for analyzing the stability of ordinary differential equations; in particular, for infinitesimally small perturbations, which is presènted as a justification for the system linearization.

In the second case a modal analysis is in general necessary. To achieve this, use is made of the Galerkin process which is based on a truncation of the modal expansion. A linearization again limits the amount of work involved. The use of these methods is wide-spread and well published. Since this approach does not constitute the subject of this thesis, reference will just be made to the works by Bolotin[4,5] and Eckhaus[6] as the most recently published books.

As distinct from these approximate methods, Lyapunov's Direct Method deals directly with the system of partial differential equations without resorting to approximation. Moreover, it is potentially applicable for the stability analysis of nonlinear systems. Thus it is not surprising that attmepts have been made to apply Lyapunov's Direct Method to derive sufficient conditions for the stability of equilibrium solutions of systems of partial differential equations. A step toward applying Lyapunov's Direct Method to partial differential equations was made by Massera[7], who extended this method to denumerably infinite systmes of ordinary differential equations.

The application of Lyapunov's Direct Method for the stability analysis of solutions to partial differential equations requires a generalization of the method to function spaces in which a metric ρ is defined. Consequently, the concepts of stability must be defined in terms of this metric. A general stability theory now based on the existence of a Lyapunov functional is established by Zubov[8] for the invariant sets of dynamical systems in general metric spaces. Zubov employs this theory in Chapter 5 to derive results for systems of partial differential equations.

The most general type of system that has been considered is of the form

$$\frac{\partial \underline{u}(t,\underline{x})}{\partial t} = \underline{L}\, \underline{u}(t,\underline{x}) \tag{I-1}$$

where $\underline{u}(t,\underline{x})$ is an n-dimensional vector valued function defined over some region Ω of an m-dimensional Euclidean space E^m. $\underline{L}$ is a linear or nonlinear matrix differential operation defined on Ω. To specify solutions, a set of boundary conditions must be given. In addition, a solution will depend on some initial function $\underline{u}_0(\underline{x})$ belonging to an n-dimensional space of initial functions.

The application of Lyapunov stability theory for the determination of stability conditions for equilibrium solutions of (I-1) is almost entirely based on the work of Zubov(8). However, in all those cases the validity of the results depends on the system being a dynamical system, i.e., on the fact that the solutions possess the group property, or to a somewhat lesser extent, the semi-group property ($t > 0$ only). The mathematical justification of this fact is either extremely vague or omitted.

Zubov(8) establishes quite conclusive results for systems of the form

$$\frac{\partial \underline{u}(t,\underline{x})}{\partial t} = \underline{f}\left(\underline{x}, \underline{u}, \frac{\partial \underline{u}}{\partial \underline{x}}\right) \tag{I-2}$$

the solutions of which constitute, under suitable assumptions, a dynamical system. Brayton and Miranker(9) apply his results to establish stability conditions for a nonlinear system representing an electrical circuit, without properly verifying the exact conditions for a dynamical system. Blodgett(10) takes a chemical reactor model to apply Zubov's results.

Zubov also compares the stability properties of the trivial solutions of the system

$$\frac{\partial u_s}{\partial t} = f_s\,(u_1,\, \ldots, u_n) + \sum_{i=1}^{k} b_i \frac{\partial u_s}{\partial x_i} \quad (s=1,2,\ldots,n) \tag{I-3}$$

and the related system of ordinary differential equations

$$\frac{du_s}{dt} = f_s\,(u_1, \ldots,\, u_n) \quad (s=1,2,\ldots,n) \tag{I-4}$$

He shows that the asymptotic stability of the trivial solution of (I-4) assures the asymptotic stability of the trivial solution of (I-3). A similar result relates the stability behavior of the equilibrium of the system of partial differential equations of higher order

$$\frac{\partial \underline{u}}{\partial t} = \sum_{\sum_{j=1}^{m} \alpha_j = 0}^{M} \underline{A}_{\alpha_1 \cdots \alpha_m} \frac{\partial^{\alpha_1 + \ldots + \alpha_m} \underline{u}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} \tag{I-5}$$

to the stability of the equilibrium of the system

$$\frac{d\,\underline{u}}{dt} = \underline{A}\,\underline{u}. \tag{I-6}$$

The nature of the derived results is very theoretical and often difficult to implement in practical applications. Hsu(11) applied Zubov's results to a nuclear reactor system, but does not verify the dynamical system properties. Wang(12) in a kind of survey paper, considers those $\underline{L}$ operators which are infinitesimal generators of semi-groups. However, in this case, the conditions are only sufficient for stability.

Another class of systems frequently encountered is of the form

$$\frac{\partial^2 \underline{u}(t,\underline{x})}{\partial t^2} + \mu \frac{\partial\, \underline{u}(t,\underline{x})}{\partial\, t} + \underline{L}\; \underline{u}(t,\underline{x}) = \underline{0} \tag{I-7}$$

with $\underline{u}(t,\underline{x})$ and $\underline{L}$ as under (I-1). Although (I-7) can be reduced to the form (I-1), it has the distinct advantage that the Lyapunov functional can readily be derived from the total system energy, again giving only sufficient conditions for stability. Most other contributions are not as general as the ones above, but reflect more direct applications to specific problems.

Movchan[13] considered the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - a\frac{\partial^2 u}{\partial x^2} = 0 \tag{I-8}$$

with the boundary conditions:

$$u = \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } x=0 \text{ and } x=1.$$

By defining the metric ρ in a suitable manner he is able to verify results from the theory of vibrations of plates by taking as Lyapunov functional:

$$V(u) = \int_0^1 (u_{xx}^2 + a\, u_x^2 + u_t^2)\, d\, x.$$

Similarly Movchan[14] verifies classical stability results for a system of hinged rectangular plates under compression, the deflection of which, $u(x,y,t)$, is given by the dimensionless equation:

$$\frac{\partial^2 u}{\partial t^2} + a_1\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial u^4}{\partial x^2 \partial y^2} + a_2\frac{\partial^4 u}{\partial y^4} - \pi^2 a_3\frac{\partial^2 u}{\partial x^2} - \pi^2 a_4\frac{\partial^2 u}{\partial y^2} = 0 \tag{I-9}$$

with boundary conditions:

$$u = \frac{\partial^2 u}{\partial x^2} = 0 \text{ at } x=0,\ x=1, \quad u = \frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = 0,\ y = 1.$$

Wang[15] studies the stability of a simplified flexible vehicle with a dimensionless equation of perturbed motion about its equilibrium state as given by

$$m(x)v_o^2\ell^2\frac{\partial^2 u(t,x)}{\partial t^2} + v_o\ell^3 k_d(t,x)\frac{\partial u(t,x)}{\partial\ t} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 u(t,x)}{\partial\ x^2}\right) = 0 \tag{I-10}$$

and boundary conditions

$$u(t,0) = 0,\ \frac{\partial u(t,x)}{\partial\ x}\Big|_{x=0} = 0\ ;\ EI(x)\frac{\partial^2 u(t,x)}{\partial\ x^2}\Big|_{x=1} = 0$$

and $\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 u(t,x)}{\partial x^2}\right)\Big|_{x=1} = 2\pi\,\rho_a\, v_0^2 \ell^2 ab\left[\frac{\partial u(t,x)}{\partial t}+\frac{\partial u(t,x)}{\partial x}\right]\Big|_{x=1}$

But he does not consider the existence problems of the solutions involved.

Parks(16) applies Lyapunov's Direct Method to the panel flutter problem. The equation in dimensionless form is given by

$$\mu\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+d\frac{\partial^4 u}{\partial x^4}-f\frac{\partial^2 u}{\partial x^2}+M\frac{\partial u}{\partial x}=0 \qquad \text{(I-11)}$$

and boundary conditions

$$u=\frac{\partial^2 u}{\partial x^2}=0 \quad \text{for } x=0 \text{ and } x=1.$$

In these applications an important role in deriving the sufficient conditions for stability is played by integral inequalities. The results obtained are presented without mathematical rigor. Another deficiency is the general interpretation of the conditions obtained for stability.

In order to discuss stability in a meaningful sense it is often necessary to put restrictions on the initial states. Although Volkov(17) implied this in an earlier work, the idea of introducing a second metric for this purpose seems to have been originated by Movchan(18). Stability is then defined in terms of the two metrics, rather than one. Wang(19) uses this concept in a stability analysis of elastic and aeroelastic systems.

The work of Lakshmikantham(20) is closely related to Zubov's results. A practical application can be found in Wei's paper(21) in which the stability of a system of partial differential equations describing the first order chemical reaction in the presence of a catalyst is analyzed. This system can be reduced to a pair of identical partial differential equations of the form

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-\phi^2\, u\,\exp\frac{\beta u(1-u)}{1+\beta(1-u)} \qquad \text{(I-12)}$$

with $0 \leq x \leq 1$ and boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \text{ and } u(t,1) = 1.$$

After linearizing (I-12) the Euclidean metric has been taken as a Lyapunov functional.

This survey of all the significant contributions indicates that many problems concerning the application of Lyapunov stability theory to partial differential equations remain unsolved. In the next section the problems involved will be analyzed in more detail, and the most logical approach to further applications will be established.

B. General Problem Areas

The literature survey shows that the main theoretical contribution to Lyapunov stability theory for solutions to partial differential equations is given by Zubov[(8)]. The important restriction on his result is the requirement that the system of partial differential equations defines a dynamical system. This implies that $\underline{L}$ of (I-1) must generate an operator which possesses the group property. However, many partial differential operators can only be defined as the infinitesimal generators of semi-groups. Thus the "group property" will only be satisfied for $t \geq 0$. Since stability is usually concerned with the properties of positive half trajectories, it seems natural that a Lyapunov stability theory can be formulated for systems having only the semi-group property. This suggests the development of a general stability theory for the class of operators generating a semi-group. This must be followed by the formulation of $\underline{L}$ in terms of this class of operators and where possible the results so obtained should be related to the actual boundary value problem.

After solving the problem for the time-invariant case there arises the possibility of extending the results to the time-invariant case.

Another extension can be directed towards the development of a Lyapunov stability theory for nonlinear partial differential equations. The formulation of a Lyapunov stability theory for semi-groups appears a natural start for research in this field. This is because the infinitesimal generators of contraction semi-groups, which are stable, are the linear dissipative operators. The natural extension of these linear dissipative operators for the nonlinear case are the monotone operators[22]. The question arises, therefore, if it is possible to extend the Lyapunov stability theory for systems with dissipative operators to include monotone operators as well.

Many of the above problems touch on research areas in mathematics which are currently being explored. It is expected that many of the above problems will provide an emerging field of future research in stability theory.

II. STATEMENT OF THE PROBLEM

A. Formal Partial Differential Equations

Many of the physical problems that are formulated by partial differential equations can be described formally by the general equation

$$\frac{\partial \underline{u}(t,\underline{x})}{\partial t} = \underline{L}\ \underline{u}(t,\underline{x}) \quad \underline{x} \in \Omega \tag{II-1}$$

where $\underline{u}(t,\underline{x})$ is an n-vector function and $\underline{L}$ is a matrix whose elements are linear or nonlinear differential operators specified on a bounded connected open subset Ω of an m-dimensional Euclidean space, E^m. The parameters of $\underline{L}$ can be space and time dependent.

In order to specify solutions to (II-1) a set of additional constraints or boundary conditions must be given by

$$\underline{H}\ \underline{u}(t,\underline{x}') = \underline{0} \quad \underline{x}' \in \partial\Omega \tag{II-2}$$

where $\underline{H}$ is a matrix whose elements are specified differential operators and $\partial\Omega$ is the boundary of Ω, $\bar{\Omega} = \Omega + \partial\Omega$.

In addition to the boundary conditions, solutions to (II-1) will depend on some initial functions $\underline{u}_0(\underline{x})$. It will be assumed for the moment that, given some initial function $\underline{u}_0(\underline{x})$ belonging to some normed linear space H, it can be shown that solutions to (II-1) and (II-2) exist and belong to H. A solution to (II-1) and (II-2) will be designated as $\underline{u}(t,\underline{x};\underline{u}_0(\underline{x}),t_0)$, that is, the solution starting at t_0 and with initial conditions $\underline{u}_0(\underline{x})$, $\underline{u}(t_0,\underline{x};\underline{u}_0(\underline{x}),t_0) = \underline{u}_0(\underline{x})$.

The solutions that are of particular interest in stability studies are the equilibrium solutions, $\underline{u}_{eq}(\underline{x})$. The equilibrium solutions can be defined as:

Definition II-1. An equilibrium solution, $\underline{u}_{eq}(\underline{x})$ is a solution of (II-1) and (II-2) such that $\frac{\partial \underline{u}_{eq}(t,\underline{x})}{\partial t} = \underline{0}$ for all $t \geq t_0$ and all $\underline{x} \varepsilon \bar{\Omega}$. Thus $\underline{u}(t,\underline{x};\underline{u}_{eq}(\underline{x}),t_0) = \underline{u}_{eq}(\underline{x})$ for all $t \geq t_0$. This is the same as determining the $\underline{u}_{eq}(\underline{x})$ such that $\underline{L}\, \underline{u}_{eq}(\underline{x}) = \underline{0}$ and $\underline{H}\, \underline{u}_{eq}(\underline{x}') = \underline{0}$, $\underline{x}' \varepsilon \partial \Omega$.

Stability can be defined in many different ways; however, stability will be defined here in the sense of Lyapunov.

Definition II-2. The equilibrium solution $\underline{u}_{eq}(\underline{x})$ of (II-1) and (II-2) is said to be stable in the sense of Lyapunov if for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for $\underline{u}_0(\underline{x}) \varepsilon H$, $||\underline{u}_0(\underline{x}) - \underline{u}_{eq}(\underline{x})|| < \delta$ implies $||\underline{u}(t,\underline{x};\underline{u}_0(\underline{x}),t_0) - \underline{u}_{eq}(\underline{x})|| < \varepsilon$ for all $t \geq t_0$.

Definition II-3. The equilibrium solution $\underline{u}_{eq}(\underline{x})$ of (II-1) and (II-2) is said to be asymptotically stable if it is stable and in addition $||\underline{u}(t,\underline{x};\underline{u}_0(\underline{x}),t_0) - \underline{u}_{eq}(\underline{x})|| \to 0$ as $t \to \infty$.

It follows from the literature survey that many authors have investigated problems similar to the one formulated above using Lyapunov's Direct Method. In many cases no attention has been paid to the question of existence of solutions to (II-1) and (II-2). Furthermore Lyapunov's Direct Method for ordinary differential equations is based on the properties of bounded operators in finite dimensional space. In solving the stability problem for (II-1) – (II-2), it is generally assumed that the functions are defined on the proper spaces, i.e., the functions possess enough continuous derivatives. It is furthermore assumed that all operations of differentiation, integration, etc., can formally be carried out and that the functions resulting from these operations still belong to the defined normed linear space, i.e., that the space is complete.

In order to satisfy all these requirements and thus conduct a rigorous investigation of the existence and stability properties of solutions to (II-1) and (II-2), the formulation of the problem must be carefully scrutinized. This investigation makes extensive use of the concepts of functional analysis, in particular, the analysis of functions defined in complete spaces. For this purpose the formal partial differential equation (II-1), that is, without boundary conditions, is considered and the properties of the formal partial differential operator are studied in spaces that possess the necessary differentiability properties. As a part of the required technique, it is necessary to complete the spaces of differentiable functions. For this purpose it is necessary to introduce suitably defined generalized derivatives.

Once this process is carried out (II-1) can be interpreted as a general operator differential equation. A Lyapunov stability theory will be developed for a certain class of operator differential equations, which is analogous to Lyapunov's Direct Method for ordinary differential equations. The formal partial differential operator is then a restriction of a differential operator from this class. Finally there remains the problem of relating the stability properties of the formal partial differential equation to those of the original boundary value problem. In the next section the general operator differential equation is evaluated.

B. Evaluation to Operator Differential Equations

Lyapunov's Direct Method for finite dimensional systems of ordinary differential equations is based on certain fundamental properties of the solutions, which all systems of finite dimensional ordinary differential equations have in common. Similar properties hold for certain classes of operator differential equations to which certain classes of formal partial differential equations belong.

For the characterization of the operator differential equation consider

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x} \tag{II-3}$$

Let (II-3) be valid for all $\underline{x} \varepsilon X$ where X is some n-dimensional Euclidean space, E^n, and $\underline{A}$ a linear operator on X into X. Thus, $\underline{A}$ is a bounded linear operator. Then (II-3) becomes a stationary linear ordinary differential equation for which a matrix representation is obtained by choosing a basis in E^n. A Lyapunov stability theory has been developed for these equations. This theory is based on the properties of bounded operators in finite dimensional spaces.

If X is a general normed linear space, then $\underline{A}$ no longer need have these properties; in fact, $\underline{A}$ might be unbounded and (II-3) should be specified as

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x} \quad (\underline{x} \varepsilon \mathcal{D}(\underline{A}) \subseteq X;\ \underline{A} : \mathcal{D}(\underline{A}) \rightarrow \mathcal{R}(\underline{A}) \subseteq X) \tag{II-4}$$

where $\mathcal{D}(\underline{A})$ is the domain of $\underline{A}$ and $\mathcal{R}(\underline{A})$ is the range of $\underline{A}$. For these unbounded operators on general function spaces, a mathematical theory has emerged in which the properties of the solutions are studied on one-parameter families of bounded linear operators, the semi-groups and groups.

Lyapunov's Direct Method for ordinary differential equations requires the construction of a Lyapunov <u>function</u>. For a general operator differential equation (II-4) this requires the construction of a Lyapunov <u>functional</u>. The stability properties follow from an evaluation of the time derivative of the Lyapunov functional along the solutions. For the Lyapunov functional the time derivative must formally be defined, and there is a possibility that it does not exist for all $\underline{x} \varepsilon X$. In the context of semi-groups this time derivative can easily be derived.

In the following it should be clear from the context whether $\underline{x}$ must be interpreted as an element of the normed linear space X (in the general theoretical development) or whether $\underline{x}$ is a space variable in E^m, the space on which the differential operator is defined.

A large class of formal linear stationary partial differential equations

$$\frac{\partial \underline{u}(t,\underline{x})}{\partial t} = \underline{L}(\underline{x})\ \underline{u}\ (t,\underline{x}) \qquad \underline{x} \in \Omega \tag{II-5}$$

where Ω is a bounded connected open subset of an m-dimensional Euclidean space, E^m, and $\underline{L}(\underline{x})$ is a formal partial differential operator on the space variable $\underline{x}$, is subsummed in a family of general operator differential equations (II-4) which generate a semi-group or a group. Thus, the operator $\underline{A}$ in (II-4) is an extension of $\underline{L}(\underline{x})$ in (II-5) and coincides with $\underline{L}(\underline{x})$ when the functions $\underline{u}$ are sufficiently smooth.

The main problem is to deduce from (II-5) the form of the space X, the linear operator $\underline{A}$, the domain $\mathcal{D}(\underline{A})$ and the range $\mathcal{R}(\underline{A})$. This in itself is not always easy.

C. A Stability Theory

In the preceding sections the difficulties of establishing a Lyapunov stability theory for partial differential equations (II-1) and (II-2) have been discussed. On the other hand, the general linear operator differential equation (II-4) can be considered as a generalization of a system of linear ordinary differential equations for which a Lyapunov stability theory exists. In order to derive a Lyapunov stability theory which can rigorously be applied to a class of partial differential equations, the stability problem will be formulated as follows:

1. Develop a Lyapunov stability theory for a class of operator differential equations (II-4) similar to Lyapunov's Direct Method for ordinary differential equations.

2. Extend the formal partial differential equations to operator equations on complete function spaces in such a way that all mathematical operations can be carried out rigorously.

3. Associate with the formal partial differential operator, so defined, a boundary value problem and give a formulation of the problem in terms of the general operator differential equation for which the stability theory is developed.

4. Give applications of the developed stability theory to specific boundary value problems.

Even though the scope of these problems is limited to a small class of partial differential equations, which are linear and stationary, they constitute one of the first developments of a mathematically rigorous approach to the stability investigations of partial differential equations.

D. Contributions to the Problem

The research concerning the stability properties of the solutions to certain classes of partial differential equations was carried out as part of a research project sponsored by the National Aeronautics and Space Administration under Grant Number NGR 39-011-039 with the University of Pittsburgh. Dr. William G. Vogt, Associate Professor of Electrical Engineering and Principal Investigator for this project originated this research effort and in the course of the research has contributed considerably to the results reported in this thesis.

The approach to the stability problem and the results obtained were developed in the course of numerous discussions between Dr. Vogt and the author. Since the results reported are to a large extent obtained as a joint effort, it is difficult to isolate many single results as principally Dr. Vogt's or the author's. Some of these are indicated below.

Most of the research carried out so far on this subject lacked a solid mathematical justification. The initial efforts by the author gave a verification, using Lyapunov stability theory, of stability results obtained by Eckhaus[(6)] who used approximate methods. These results were eventually established rigorously for the linear case (the case for which the results of Eckhaus are certain to hold) and are given in Chapter IX.

The important contribution at this stage was the introduction of the concept of equivalent inner products by the author. However, its use was primarily aimed at obtaining self-adjointness properties for the formal operator involved and improving the estimates obtained by the use of integral inequalities. The abstraction of this notion and its final implementation in a stability theory for dissipative operators is largely due to Dr. Vogt.

Lyapunov's Direct Method for systems of ordinary differential equations is based on some fundamental properties, i.e., group properties, of the solutions. A first observation showed that this group property is not limited to finite dimensional systems of ordinary differential equations, but is common to a larger class of systems. Moreover, a still larger class of systems possesses the more general semi-group property. This led to a formulation of a Lyapunov stability theory for groups and semi-groups. The stability properties are in this case directly coupled to the trajectory structure, i.e., the solutions. The main goal became to express the conditions for the stability of the solutions in terms of the operator generating the group or semi-group--in particular, those generating contraction groups or semi-groups. A large class of such operators is formed by the bounded and unbounded dissipative operators.

Dissipativity, in a particular case, is an inner product property while stability is a norm property. This implies a Hilbert space theory. Since the stability properties are invariant under equivalent norms, the introduction of the principle of equivalent inner products mentioned above is crucial to link the dissipativity property to the stability properties. Two inner products are equivalent if and only if their induced norms are equivalent.

This subsequently allows the formulation of sufficient conditions on an operator to generate stable or asymptotically stable semi-groups and necessary and sufficient conditions on an operator to generate stable or asymptotically stable groups. Dr. Vogt proved an important norm property of groups which is used in the proof of the last statement.

The semi-group (group) structure enables one to formulate an operator differential equation with its infinitesimal generator. The derived stability conditions extend to the stability of the null solutions of this differential equation. The stability theorems are given in Chapter V.

The conditions to be imposed on the operator do not restrict the class of operators to bounded operators. Thus a stability theory is established for a large class of operator differential equations defined in a Hilbert space.

The second stage is to formulate partial differential equations in the framework of the operator differential equations. A kind of synthesis procedure is used here.

Starting out with a formal partial differential operator, the first requirement is that the extended operator be defined in a complete space. This is done by introducing distributions. The next step is to associate boundary conditions with the formal partial differential operator. In the context of the thesis this is limited to the so-called Dirichlet boundary conditions, which are important for many stability investigations.

The class of formal partial differential operators is limited to the strongly elliptic partial differential operators. The domain and range of the extended operator are subsequently formulated in terms of Sobolev spaces. In Chapter VII, the stability theory is applied to a class of evolution equations with strongly elliptic partial differential operators and Dirichlet boundary conditions.

A similar formulation of a class of wave equations is presented in Chapter VIII. The development of a Lyapunov functional for this class of wave equations is considered one of the main contributions of this thesis. It establishes conditions for asymptotic stability of the null solution and the group property of the solutions to the wave equations.

Chapter IX is devoted to a number of applications which illustrate various aspects of the developed stability theory. With the exception of Chapter X, almost entirely the author's work, where some nonlinear systems are formally analyzed, the emphasis has been on a mathematically rigorous formulation of the stability problem. Even though only the stability properties for a class of linear, stationary, partial differential equations are established here, the directions to be pursued for enlarging this class have been opened to further research. These results appear to be a significant contribution toward a rigorous Lyapunov stability theory for a more general class of partial differential equations including nonlinear partial differential equations.

III. MATHEMATICAL PRELIMINARIES

Engineers have become familiar with the concepts of linear vector spaces and linear transformations on these spaces through the introduction of the state space approach in control theory and the development of a Lyapunov stability theory for ordinary differential equations. The stability study of general operator differential equations adds another dimension to these concepts. Functional analysis is a fundamental tool in the study of operators defined in general function spaces. In the following sections some of the basic notions and properties that are important in the stability analysis of operator differential equations will be introduced. A more detailed treatment and examples can be found in any book on functional analysis, for example, Reference 23.

A. Normed Linear Spaces

In the previous chapter the general operator differential equation was introduced by defining $\underline{A}$ as an operator in a normed linear space. Such a space is defined as follows:

Definition III-1. Let X be a vector space over the field of real or complex numbers. A norm on X, denoted by $||\cdot||$, is a real-valued function on X with the following properties:

i. $||\underline{x}|| \geq 0$ for all $\underline{x} \varepsilon X$.

ii. $\underline{x} \neq \underline{0}$ implies $||\underline{x}|| \neq 0$.

iii. $||\alpha \underline{x}|| = |\alpha| \, ||\underline{x}||$, α some real or complex scalar.

iv. $||\underline{x} + \underline{y}|| \leq ||\underline{x}|| + ||\underline{y}||$ (triangle inequality).

The vector space X, together with a norm on X, is called a normed linear space. When the scalars over X are the reals, X is called a real normed linear space.

The finite dimensional real Euclidean space, R^n, can be made into a normed linear space by defining the norm by

$$||\underline{x}|| = (\sum_{i=1}^{n} x_i^2)^{1/2} \quad \forall\ \underline{x} \in R^n.$$

The norm for a space can be defined in more than one way. Thus another norm for R^n is defined by

$$||\underline{x}|| = \sup_i |x_i| \quad \forall\ \underline{x} \in R^n.$$

These two norms create two different normed linear spaces, with possibly different basic properties. The introduction of a norm for R^n makes (II-3) a special case of the general linear operator differential equations as defined by (II-4).

The norm of X induces a metric or distance d which is defined by

$$d(\underline{x},\underline{y}) = ||\underline{x} - \underline{y}|| \quad \underline{x},\ \underline{y} \in X$$

i.e., the distance between two elements $\underline{x}$ and $\underline{y} \in X$ is given by $||\underline{x} - \underline{y}||$. According to their properties one can distinguish different classes of normed linear spaces. First of all the pre-Hilbert spaces are defined:

Definitions III-2. A real or complex normed linear space X is called a pre-Hilbert space if its norm satisfies the condition

$$||\underline{x} + \underline{y}||^2 + ||\underline{x} - \underline{y}||^2 = 2(||\underline{x}||^2 + ||\underline{y}||^2).$$

The more important normed linear spaces are the complete normed linear spaces or Banach spaces. A Banach space X is a normed linear space in which every Cauchy sequence converges with respect to the norm to a limit point in X. A complete pre-Hilbert space is called a Hilbert space. The norm in the Hilbert space is the one induced by an inner product, $||\underline{x}|| = \langle\underline{x},\underline{x}\rangle^{1/2}$. The inner product, $\langle\cdot,\cdot\rangle$, is defined in a real pre-Hilbert space by:

$$\langle\underline{x},\underline{y}\rangle = \frac{1}{4}(||\underline{x} + \underline{y}||^2 - ||\underline{x} - \underline{y}||^2).$$

and in a complex pre-Hilbert space by:

$$<\underline{x},\underline{y}> = <\underline{x},\underline{y}>_1 + i<\underline{x},i\underline{y}>_1$$

where $$<\underline{x},\underline{y}>_1 = \frac{1}{4}(||\underline{x}+\underline{y}||^2 - ||\underline{x}-\underline{y}||^2).$$

This inner product has the following properties:

i. $<\alpha\, \underline{x},\underline{y}> = \alpha <\underline{x},\underline{y}>$ α real or complex

ii. $<\underline{x}+\underline{y},\underline{z}> = <\underline{x},\underline{z}> + <\underline{y},\underline{z}>$

iii. $<\underline{x},\underline{y}> = <\underline{y},\underline{x}>$ for real space ($= \overline{<\underline{y},\underline{x}>}$ for complex space)

iv. $<\underline{x},\underline{x}> > 0$ whenever $\underline{x} \neq 0$.

The finite dimensional real Euclidean space R^n, is made into a Hilbert space by defining the inner product by the following finite series:

$$<\underline{x},\underline{y}> = x_1y_1 + x_2y_2 + \ldots + x_ny_n.$$

Stability is defined in terms of the norm chosen for the linear space. It might be expected that the stability properties will depend on the particular norm selected for the space. The norm can in general be chosen in many different ways. The stability properties are preserved from one space to another if the norms are equivalent. Two norms, $||\cdot||_1$ and $||\cdot||_2$, are equivalent if there exist constants c_1 and c_2, $\infty>c_2>c_1>0$ such that

$$c_1\,||\underline{x}||_1 \leqq ||\underline{x}||_2 \leqq c_2||\underline{x}||_1 \qquad (\forall \underline{x}\varepsilon X) \qquad \text{(III-1)}$$

Moreover the spaces X_1 and X_2 denoted by $X_1 = (X;\ ||\cdot||_1)$ and $X_2 = (X;||\cdot||_2)$ respectively are topologically equivalent.

For systems described on finite dimensional Euclidean spaces all norms are equivalent. Thus stability with respect to one norm implies stability with respect to another, i.e., stability in one normed space implies stability in another. The stability problem evolves basically around the selection of a norm, i.e., the normed linear space. For Hilbert spaces this reduces to the selection of an inner product.

Since the stability theory to be developed concerns primarily dissipative operators[24], which are defined in terms of the inner product, Hilbert spaces are very important. Dissipative operators can be studied in somewhat more general spaces. These so-called semi-inner product spaces were introduced by Lumer[25]. Lumer and Phillips[26] studied dissipative operators in these particular spaces.

Definition III-3. A semi-inner product space is defined on a complex or real vector space X with norm $||\cdot||$ if to each pair $\underline{x},\underline{y} \in X$ there corresponds a complex or real number $[\underline{x},\underline{y}]$ such that

i. $[\underline{x} + \underline{y},\underline{z}] = [\underline{x},\underline{z}] + [\underline{y},\underline{z}]$

ii. $[\alpha\underline{x},\underline{y}] = \alpha[\underline{x},\underline{y}]$

iii. $[\underline{x},\underline{x}] = ||x||^2$ for $\underline{x} \neq \underline{0}$

iv. $|[\underline{x},\underline{y}]|^2 \leq [\underline{x},\underline{x}] \cdot [\underline{y},\underline{y}]$

Any Banach space can be made into a semi-inner product space. In particular, for a Hilbert space the only semi-inner product is the usual inner product.

The relevant properties of normed linear spaces can be found in any book on functional analysis and the papers given as references. Where necessary in the following chapters, these properties will be recalled. In the next section, linear operators and their properties in these various spaces are introduced.

B. Linear Operators

This section is devoted to the characterization of operators[23],[27]. Let X and Y be two linear spaces over the same real or complex field. The objective is to characterize the mappings $\underline{T}: \underline{x} \rightarrow \underline{y}$ where $\underline{x} \in X$ and $\underline{y} \in Y$.

Definition III-4. The mapping $\underline{T}: \underline{x} \to \underline{y} = \underline{T}(\underline{x})$ defined on a linear manifold D of X and taking values in Y with the property $\underline{T}(\alpha\, \underline{x}_1 + \beta\, x_2) = \alpha(\underline{Tx}_1) + \beta(\underline{Tx}_2)$ is called a linear operator on $D \subseteq X$ into Y.

$D = \mathcal{D}(\underline{T})$, the set in X on which $\underline{T}$ operates, is called the domain of $\underline{T}$. The set in Y which results from the operation of $\underline{T}$ on $\mathcal{D}(\underline{T}) \subseteq X$ is called the range of $\underline{T}$, $\mathcal{R}(\underline{T})$,

$$\mathcal{R}(\underline{T}) = \{\underline{y} \in Y \ni \underline{y} = \underline{T}\,\underline{x},\ \underline{x} \in \mathcal{D}(\underline{T})\}.$$

In certain applications the null space $\mathcal{N}(\underline{T})$ is used as defined by

$$\mathcal{N}(\underline{T}) = \{\underline{x} \in \mathcal{D}(\underline{T}) \ni \underline{T}\,\underline{x} = 0\}.$$

If the range $\mathcal{R}(\underline{T})$ is contained in the scalar field K, then $\underline{T}$ is called a linear functional. If a linear operator $\underline{T}$ gives a one-to-one map of $\mathcal{D}(\underline{T})$ onto $\mathcal{R}(T)$, then the inverse map $\underline{T}^{-1}$ gives a linear operator on $\mathcal{R}(\underline{T})$ onto $\mathcal{D}(\underline{T})$:

$$\underline{T}^{-1}\,\underline{T}\,\underline{x} = \underline{x} \text{ for } \underline{x} \in \mathcal{D}(\underline{T}) \text{ and } \underline{T}\,\underline{T}^{-1}\,\underline{y} = \underline{y} \text{ for } \underline{y} \in \mathcal{R}(\underline{T}).$$

$\underline{T}^{-1}$ is the inverse of $\underline{T}$.

Proposition III-1. A linear operator $\underline{T}$ admits the inverse $\underline{T}^{-1}$ if and only if $\underline{T}\,\underline{x} = 0$ implies $\underline{x} = \underline{0}$.

Definition III-5. Let $\underline{T}_1$ and $\underline{T}_2$ be linear operators with domains $\mathcal{D}(\underline{T}_1)$ and $\mathcal{D}(\underline{T}_2)$ both contained in a linear space X, and ranges $\mathcal{R}(\underline{T}_1)$ and $\mathcal{R}(\underline{T}_2)$ both contained in a linear space Y. Then $\underline{T}_1 = \underline{T}_2$ if and only if $\mathcal{D}(\underline{T}_1) = \mathcal{D}(\underline{T}_2)$ and $\underline{T}_1\,\underline{x} = \underline{T}_2\,\underline{x}$ for all $\underline{x} \in \mathcal{D}(\underline{T}_1) = \mathcal{D}(\underline{T}_2)$. If $\mathcal{D}(\underline{T}_1) \subseteq \mathcal{D}(\underline{T}_2)$ and $\underline{T}_1\underline{x} = \underline{T}_2\underline{x}$ for all $\underline{x} \in \mathcal{D}(\underline{T}_1)$, then $\underline{T}_2$ is called an extension of $\underline{T}_1$ and $\underline{T}_1$ a restriction of $\underline{T}_2$, written as $\underline{T}_1 \subseteq \underline{T}_2$.

An important role is played by the bounded linear operators. Let X and Y be normed linear spaces. A linear operator $\underline{T}$ with domain in X and range in Y is bounded if there exists a positive constant $M < \infty$ such that for all $\underline{x} \in \mathcal{D}(\underline{T})$

$$||\underline{T}\,\underline{x}|| \leqq M||\underline{x}||.$$

$\underline{T}$ is continuous at a point $\underline{x} \in X$ and $\underline{T}$ is uniformly continuous in X are equivalent statements. The class of all bounded linear operators on X into Y is designated as L(X,Y) i.e., $\underline{T} \in L(X,Y)$ $\mathcal{D}(\underline{T}) \equiv X$, $\mathcal{R}(\underline{T}) \subseteq Y$, $\underline{T}$ is bounded.

Proposition III-2. Let X and Y be normed linear spaces. Then a linear operator on $\mathcal{D}(\underline{T}) \subseteq X$ into Y admits a continuous inverse $\underline{T}^{-1}$ if and only if there exists a positive constant γ such that

$$||\underline{T}\,\underline{x}|| \geqq \gamma\,||x|| \quad \text{for every } \underline{x} \in \mathcal{D}(\underline{T}).$$

Definition III-6. If $\underline{T}$ is a bounded linear operator on a normed linear space X into a normed linear space Y, then its norm is defined by

$$||\underline{T}|| = \sup_{||\underline{x}||=1} ||\underline{T}\,\underline{x}|| = \sup_{\underline{x}\neq\underline{0}} \frac{||\underline{T}\,\underline{x}||}{||\underline{x}||}$$

An extension of the notion of a bounded linear operator is that of a closed linear operator. The definition is based on the notion of graph of $\underline{T}$, $G(\underline{T})$.

Definition III-7. The product space X x Y is defined as the normed linear space of all ordered pairs $(\underline{x},\underline{y})$, $\underline{x} \in X$, $\underline{y} \in Y$, with the usual definitions of addition and scalar multiplication and with norm given by

$$||(\underline{x},\underline{y})|| = \max\ \{||\underline{x}||,\ ||\underline{y}||\}.$$

Definition III-8. The graph $G(\underline{T})$ of $\underline{T}$ is the set $\{(\underline{x},\underline{Tx})\,|\,x \in \mathcal{D}(\underline{T})\}$. Since $\underline{T}$ is linear, $G(\underline{T})$ is a subspace of X x Y. If the graph of $\underline{T}$ is closed in X x Y, then $\underline{T}$ is said to be closed in X. When there is no ambiguity concerning the space X, $\underline{T}$ is said to be closed.

The following remarks can be made:

i. $\underline{T}$ is closed if and only if $\{\underline{x}_n\}$ in $\mathcal{D}(\underline{T})$, $\underline{x}_n \to \underline{x}$, $\underline{T}\,\underline{x}_n \to \underline{y}$, imply $\underline{x} \in \mathcal{D}(\underline{T})$ and $\underline{T}\,\underline{x} = \underline{y}$.

ii. If $\underline{T}$ is 1-1 and closed, then $\underline{T}^{-1}$ is closed.

iii. The null space of a closed operator is closed.

iv. If $\mathcal{D}(\underline{T})$ is closed and $\underline{T}$ is continuous, then $\underline{T}$ is closed.

v. The continuity of $\underline{T}$ does not necessarily imply that $\underline{T}$ is closed.

$\underline{T}$ is closed does not necessarily imply that $\underline{T}$ is continuous.

Closed-Graph Theorem. A closed linear operator mapping a Banach space into a Banach space is continuous.

Some additional properties of closed operators are introduced after defining the adjoint operator.

For a Banach space X the conjugate or dual space, the Banach space of bounded linear functionals on X, is indicated by X'. Let $\underline{T}$ be a linear operator mapping the normed linear space X into the normed linear space Y and with domain dense in X. Then the conjugate of $\underline{T}$ is denoted by $\underline{T}'$. The definition can be found in [23].

If X and Y are Hilbert spaces, then the notion of conjugate operator of $\underline{T}$ can be extended to that of adjoint operator of $\underline{T}$ which is denoted by $\underline{T}^*$. The operators one generally deals with are mappings from subsets of a Hilbert space X into X. Let $\underline{T}$ be such a linear operator. The adjoint $\underline{T}^*$ of T with respect to X is defined by

$$< \underline{T}\ \underline{x}, \underline{y}> = < \underline{x}, \underline{T}^*\underline{y}> \text{ for } \underline{x} \in \mathcal{D}(\underline{T}) \text{ and } \underline{y} \in \mathcal{D}(\underline{T}^*)$$

$\underline{T}^*$ exists if and only if $\mathcal{D}(\underline{T})$ is dense in X. The closure of $\mathcal{D}(\underline{T})$ in X is denoted by $\overline{\mathcal{D}(\underline{T})}$, thus $\mathcal{D}(\underline{T})$ is dense in X implies $\overline{\mathcal{D}(\underline{T})} = X$. A linear operator $\underline{T}$ on $\mathcal{D}(\underline{T}) \subseteq X$ into X will be called symmetric if $\underline{T}^* \supseteq \underline{T}$, i.e., if $\underline{T}^*$ is an extension of $\underline{T}$. A linear operator $\underline{T}$: $\mathcal{D}(\underline{T}) \to X$ is called self-adjoint if $\underline{T} = \underline{T}^*$.

The following properties of symmetric and self-adjoint operators should be noted:

i. A symmetric operator $\underline{T}$ has a closed symmetric extension

$\underline{T}^{**} = (\underline{T}^{*})^{*} \supseteq \underline{T}$.

ii. An everywhere defined symmetric operator is bounded and self-adjoint.

iii. A self-adjoint operator is closed since an adjoint operator is closed.

The relation between closed operators and their adjoints is expressed in the following important theorem and corollary:

Theorem III-1. Let $\underline{T}$ be a linear operator on $\mathcal{D}(\underline{T}) \subseteq X$ into X such that $\overline{\mathcal{D}(\underline{T})} = X$. Then $\underline{T}$ admits a closed linear extension if and only if $\underline{T}^{**} = (\underline{T}^{*})^{*}$ exists, i.e., if and only if $\overline{\mathcal{D}(\underline{T}^{*})} = X$.

Corollary III-1. If $\overline{\mathcal{D}(\underline{T})} = X$, then $\underline{T}$ is a closed linear operator if and only if $\underline{T} = \underline{T}^{**}$.

Let us next illustrate the concept of bounded and unbounded linear operators on finite and infinite dimensional spaces.

Example III-1. a. Consider the following operators in R^n.

If $\underline{x} = \text{col}\ (x_1, x_2, x_3, \ldots, x_n)$ then let

$$\underline{D}\ \underline{x} = \text{col}\ (x_1, \frac{1}{2} x_2, \ldots \frac{1}{n} x_n)$$

$$\underline{E}\ \underline{x} = \text{col}\ (x_1, 2x_2, \ldots n\ x_n).$$

If $||\underline{x}|| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ then both $\underline{D}$ and $\underline{E}$ are bounded since $||\underline{D}\ \underline{x}|| \leqq ||\underline{x}||$ and $||\underline{E}\ \underline{x}|| \leqq n||\underline{x}||$. In fact, all stationary linear operators on a finite dimensional Euclidean space are bounded.

b. Consider next the same operators on a infinite dimensional

Euclidean space R^∞, thus if $\underline{x}$ = col $(x_1, x_2, \ldots.)$ then

$$\underline{D}\ \underline{x} = \text{col}\ (x_1, \tfrac{1}{2}x_2 \ldots..)$$

$$\underline{E}\ \underline{x} = \text{col}\ (x_1, 2x_2, \ldots..).$$

If the norm of $\underline{x} \in R^\infty$ is defined by $||\underline{x}|| = (\sum_{i=1}^{\infty} x_i^2)^{1/2}$, then D will still be bounded, since $||\underline{D}\ \underline{x}|| \leqq ||\underline{x}||$. however, $\underline{E}$ will no longer be bounded.

The concept of dissipative operator is defined next.

Definition III-9. Let $\underline{T}$ be a linear operator such that $\mathcal{D}(\underline{T})$ and $\mathcal{R}(\underline{T})$ belong both to the Hilbert space X. Then $\underline{T}$ is called dissipative if

$$\text{Re} < \underline{T}\ \underline{x}, \underline{x}> \leqq 0 \quad \text{for all } \underline{x} \in \mathcal{D}(\underline{T}).$$

Definition III-10. Let $\underline{T}$ be a linear operator such that $\mathcal{D}(\underline{T})$ and $\mathcal{R}(\underline{T})$ both belong to X. Then $\underline{T}$ is called strictly dissipative if there exists a constant $c > 0$ such that

$$\text{Re} < \underline{T}\ \underline{x}, \underline{x}> \leqq - c\ ||\underline{x}||^2 \text{ for all } \underline{x} \in \mathcal{D}(\underline{T}).$$

This concept can similarly be defined in terms of semi-inner product. This concept is used frequently in the following chapters. Definitions III-9 and III-10 imply that $-\underline{T}$ is positive and positive definite respectively.

Spectral theory for an operator $\underline{T}$ is the distribution of the values of λ for which $\underline{T}_\lambda = \lambda\underline{I} - \underline{T}$ has an inverse and the properties of the inverse when it exists. The following are the definitions of the frequently used resolvent and spectrum.

Definition III-11. If λ_o is such that the range $\mathcal{R}(\underline{T}_{\lambda_o})$ is dense in X and $\underline{T}_{\lambda_o}$ has a continuous inverse $(\lambda_o \underline{I} - \underline{T})^{-1}$, then λ_o is said to be in the resolvent set $\rho(\underline{T})$ of $\underline{T}$ and this inverse $(\lambda_o\underline{I} - \underline{T})^{-1}$ is denoted by $R(\lambda_o; \underline{T})$ and it is called the resolvent (at λ_o) of $\underline{T}$. All complex numbers λ not in $\rho(\underline{T})$ form a set $\sigma(\underline{T})$ called the spectrum of $\underline{T}$. The spectrum $\sigma(\underline{T})$ is decomposed into disjoint sets $P_\sigma(\underline{T})$, $C_\sigma(\underline{T})$ and $R_\sigma(\underline{T})$ with the following properties: $P_\sigma(\underline{T})$ is the totality of complex numbers λ for which $\underline{T}_\lambda$ does not have an inverse; $P_\sigma(\underline{T})$ is called the point spectrum of $\underline{T}$.

$C_\sigma(\underline{T})$ is the totality of complex numbers λ for which $\underline{T}_\lambda$ has a discontinuous inverse with domain dense in X; $C_\sigma(\underline{T})$ is called the continuous spectrum of $\underline{T}$.

$R_\sigma(\underline{T})$ is the totality of complex numbers λ for which $\underline{T}_\lambda$ has an inverse whose domain is not dense in X; $R_\sigma(\underline{T})$ is called the residual spectrum of $\underline{T}$.

An important theorem concerning the resolvent is:

Theorem III-2. Let X be a Banach space and $\underline{T}$ a closed linear operator with its domain $\mathcal{D}(\underline{T})$ and range $\mathcal{R}(\underline{T})$ both in X. Then, for any $\lambda_o \varepsilon\ \rho(\underline{T})$ the resolvent $(\lambda_o\underline{I} - \underline{T})^{-1}$ is an everywhere defined continuous linear operator.

The following section is devoted to some properties of Hilbert spaces, which are very important for the development of the stability theory for a class of operator differential equations.

C. Hilbert Spaces

In Chapter II the general operator differential equation (II-4) was introduced with $\underline{A}$ defined on a general normed linear space X. It is next assumed that X is a Hilbert space. Thus, either the norm of X is induced by an inner product, or the norm satisfies the parallelogram law and therefore induces an inner product. In Section A equivalent norms in Banach spaces were defined. Since a Hilbert space belongs to a special class of Banach spaces, the concept of equivalent norm not only holds, but can be formulated more specifically with respect to the inner product structure of the Hilbert space. In particular, it allows the introduction of the concept of equivalent inner products as defined by:

<u>Definition III-12</u>. Let $H_1 = (H, \langle\cdot,\cdot\rangle_1)$ and $H_2 = (H, \langle\cdot,\cdot\rangle_2)$ be Hilbert spaces consisting of the elements of a linear vector space H and the inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, respectively. The inner products are called equivalent if and only if the induced norms are equivalent.

This concept of equivalent inner product enables one to "carry" the stability properties from one Hilbert space into another Hilbert space defined for the elements of a linear vector space. It is also possible that the norm induced by the inner product of a Hilbert space, $H_1 = (H; \langle\cdot,\cdot\rangle_1)$, is equivalent with another norm which does not satisfy the parallelogram law. This last norm will not induce an inner product; the resulting space $(H; ||\cdot||_2)$ is thus not a Hilbert space, but still a Banach space.

The characterization of equivalent inner products can be made more explicit by the Lax-Milgram Theorem[23].

<u>Theorem III-3</u>. (Lax-Milgram). Let H be a Hilbert space and let $B(\underline{x},\underline{y})$ be a complex-valued functional defined on the product Hilbert space H x H which satisfies the conditions:

i. Sesqui-linearity ,i.e.,

$$B(\alpha_1\underline{x}_1 + \alpha_2\underline{x}_2\ \underline{y}) = \alpha_1\ B(\underline{x}_1, \underline{y}) + \alpha_2 B(\underline{x}_2, \underline{y})$$

and

$$B(\underline{x}, \beta_1\underline{y}_1 + \beta_2\underline{y}_2) = \bar{\beta}_1 B(\underline{x},\underline{y}_1) + \bar{\beta}_2 B(\underline{x},\underline{y}_2)$$

ii. <u>Boundedness</u>, i.e., there exists a positive constant γ such that

$$|B(\underline{x},\underline{y})| \leq \gamma\ ||\underline{x}|| \cdot ||\underline{y}||.$$

iii. <u>Positivity</u>, i.e., there exists a positive constant δ such that

$$B(\underline{x},\underline{x}) \geq \delta\ ||\underline{x}||^2$$

then there exists a uniquely determined bounded linear operator $\underline{S}\ \varepsilon L(H,H)$ with a bounded linear inverse $\underline{S}^{-1}\ \varepsilon\ L(H,H)$ such that $\langle \underline{x},\underline{y}\rangle = B(x, \underline{S}\ \underline{y})$ whenever

$\underline{x}$ and $\underline{y} \in H$ and $||\underline{S}|| \leqq \delta^{-1}$, $||\underline{S}^{-1}|| \leqq \gamma$.

The implications of this theorem for the equivalent inner products are given by the following theorem:

Theorem III-4. Two inner products defined on a real linear vector space H are equivalent if and only if there exists a symmetric bounded positive definite linear operator $\underline{S} \in L(H,H)$ such that $<\underline{x},\underline{y}>_2 = <\underline{x},\underline{S}\,\underline{y}>_1$ for all $\underline{x}$, $\underline{y} \in H$.

Proof: Let $B(\underline{x},\underline{y}) = <\underline{x},\underline{S}\,\underline{y}>_1$, where $\underline{S}$ is symmetric. Then $B(\underline{x},\underline{y})$ is bilinear and $B(\underline{y},\underline{x}) = <\underline{y},\underline{S}\,\underline{x}>_1 = <\underline{x},\underline{S}\,\underline{y}>_1$. Since $\underline{S}$ is positive definite $B(\underline{x},\underline{x}) = <\underline{x},\underline{S}\,\underline{x}>_1 \geqq \delta <\underline{x},\underline{x}>_1$ and since $\underline{S}$ is bounded, $B(\underline{x},\underline{x}) = <\underline{x},\underline{S}\,\underline{x}>_1 \leqq ||\underline{x}||_1 \cdot ||\underline{S}\,\underline{x}||_1 \leqq \gamma||\underline{x}||_1^2$ there follows that $B(\underline{x},\underline{y}) = <\underline{x},\underline{y}>_2$ satisfies all the properties of an inner product. The fact that the norms induced by $<\cdot,\cdot>_1$ and $<\cdot,\cdot>_2$ are equivalent implies that the inner products are equivalent.

Suppose $<\cdot,\cdot>_1$ and $<\cdot,\cdot>_2$ are inner products with equivalent norms, i.e., there exist positive constants α and β, $0 < \alpha < \beta < \infty$, such that,

$$\alpha\, ||\cdot||_1 \leqq ||\cdot||_2 \leqq \beta||\cdot||_1 \ .$$

Then since $<\cdot,\cdot>_2$ is bilinear in real H, there follows

$$|<\underline{x},\underline{y}>_2| \leqq ||\underline{x}||_2||\underline{y}||_2 \leqq \beta^2||\underline{x}||_1||\underline{y}||_1,$$

and therefore by Theorem III-3 there exists a uniquely determined $\underline{S}$ such that

$$<\underline{x},\underline{y}>_1 = <\underline{x},\underline{S}\,\underline{y}>_2.$$

But since $<\underline{x},\underline{y}>_1 = <\underline{y},\underline{x}>_1$ there follows $<\underline{x},\underline{S}\,\underline{y}>_2 = <\underline{y},\underline{S}\,\underline{x}>_2$ and $\underline{S}$ is symmetric thus proving the theorem. $\underline{S}$ is obviously a self-adjoint operator.

In this framework of equivalent inner products it is possible to derive the stability properties of the general operator differential equation of type (II-4) from the knowledge of the properties of the operator $\underline{A}$ only. Once the formal partial differential equations are formulated in terms of the general operator differential equations, the equivalent inner products enable one to derive the maximal system parameter ranges for stability. This concept can be extended to complex Hilbert spaces.

IV. SEMI-GROUPS, GROUPS AND INFINITESIMAL GENERATORS

In order to develop a stability theory for the solutions of the operator differential equation

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x} \quad (\underline{x}\ \varepsilon \mathcal{D}(\underline{A}) \subseteq X;\ \underline{A}: \mathcal{D}(\underline{A}) \rightarrow \mathcal{R}(\underline{A}) \subseteq X) \tag{IV-1}$$

where X is a Banach space, $\underline{A}$ will be restricted to the class of operators which are the infinitesimal generators of bounded linear operator valued functions $\underline{T}_t$, $t \geqq 0$ that satisfy the condition

$$\underline{T}_{t+s} = \underline{T}_t \cdot \underline{T}_s,\ \underline{T}_0 = \underline{I}\ . \tag{IV-2}$$

The notion of infinitesimal generator, to be formulated later, was introduced by Hille and Yosida. Much about this basic concept can be found in their respective books(28) (23), together with the general theory of linear operators in function spaces. Instead of the bounded linear operators $\underline{T}_t$ having merely the semi-group property, the more restricted semi-groups of class (C_0) are introduced. It is in terms of these bounded operators that the properties of a system (IV-1) can be studied more easily than in terms of the operator $\underline{A}$.

In the following two sections the semi-groups of class (C_0), groups of class (C_0) and their infinitesimal generators are introduced.

A. Semi-Groups and Groups of Class (C_0)

Let X be a Banach space with norm $||\cdot||$. For each fixed $t \geqq 0$, let $\underline{T}_t$ be a bounded linear operator on X into X, $\underline{T}_t\ \varepsilon L(X,X)$, the set of all bounded linear operators mapping X into X. Then the single parameter family of operators $\{\underline{T}_t;\ t \geqq 0\} \subseteq L(X,X)$ with parameter $t\ \varepsilon R^+ = [0, \infty)$ is said to satisfy the <u>semi-group property</u> if

$$\underline{T}_t \cdot \underline{T}_s = \underline{T}_{t+s} \quad (t,s > 0).$$

Definition IV-1. If $\{\underline{T}_t;\ t \geqq 0\} \subseteqq L(X,X)$ satisfies the conditions

i. $\underline{T}_t \cdot \underline{T}_s = \underline{T}_{t+s} \quad (t,s \geqq 0)$

ii. $\underline{T}_0 = \underline{I}$ ($\underline{I}$ is the identify operator in $L(X,X)$)

iii. $\lim_{t \to t_0} ||\underline{T}_t \underline{x} - \underline{T}_{t_0} \underline{x}|| = 0$ for each $t_0 > 0$ and all $\underline{x}$ εX,

then $\{\underline{T}_t\}$ is called a semi-group of class (C_0).

Because of the semi-group structure[23] it follows that for a semi-group $\{\underline{T}_t\}$ of class (C_0) there exist constants $M > 0$ and $\beta < \infty$ such that

$$||\underline{T}_t|| \leqq M\, e^{\beta t} \quad (t \geqq 0)\ . \qquad \text{(IV-3)}$$

If in addition to (i), (ii), and (iii) of Definition (IV-1) β in (IV-3) can be chosen as $\beta = 0$, then

$$||\underline{T}_t|| \leqq M \qquad (0 \leqq t < \infty) \qquad \text{(IV-4)}$$

and $\{\underline{T}_t\}$ is an equi-bounded semi-group of class (C_0). In particular if $M = 1$, i.e., if

$$||\underline{T}_t|| \leqq 1 \ \text{ for } \ (0 \leqq t < \infty) \qquad \text{(IV-5)}$$

then $\{\underline{T}_t\}$ is called contraction semi-group of class (C_0).

An equi-bounded semi-group of class (C_0) is equi-continuous in t[23].

The equi-continuous semi-groups are of main interest in the following discussion of stability properties of unbounded differential operators. It is clear that since $\{\underline{T}_t\}$ is only defined for $t \geqq 0$ the corresponding operator differential equation does not generate a dynamical system. The system becomes a dynamical system if $\{\underline{T}_t\}$ can be extended to the negative time axis and $\{\underline{T}_t\}$ becomes a group:

Definition IV-2. If $\{\underline{T}_t\ ;\ -\infty < t < \infty\} \subseteqq L(X,X)$ satisfies the conditions:

i. $\underline{T}_t \cdot \underline{T}_s = \underline{T}_{t+s} \qquad (-\infty < t,s < \infty)$

ii. $\underline{T}_0 = \underline{I}$

iii. $\lim_{t \to t_0} ||\underline{T}_t \underline{x} - \underline{T}_{t_0} \underline{x}|| = 0$ for each $t_0 \varepsilon (-\infty, +\infty)$ and

for all $\underline{x} \varepsilon X$, then $\{\underline{T}_t\}$ is called a group of class (C_0).

The alternative definition (Yosida[23]) has the group $\{\underline{S}_t\}$ defined in terms of equi-continuous semi-groups $\{\underline{T}_t\}$ and $\{\hat{\underline{T}}_t\}$ by letting $\underline{S}_t = \underline{T}_t$ for $t \geqq 0$ and $\underline{S}_{-t} = \hat{\underline{T}}_t$ for $t \geqq 0$, yielding $\underline{S}_t \underline{S}_{-t} = \underline{I} = \underline{T}_t \hat{\underline{T}}_t$. Thus

$$\hat{\underline{T}}_t = \underline{T}_t^{-1} \qquad \text{(IV-6)}$$

with $\hat{\underline{T}}_t$ defined on $R(\underline{T}_t) \subseteqq X$.

The disadvantage of defining the group $\{\underline{S}_t\}$ in this manner is that $\hat{\underline{T}}_t$ as defined by (IV-6) is a semi-group only if it is defined on all of X, thus if $\overline{R(\underline{T}_t)} = X$.

If $\{\underline{T}_t\}$ is a semi-group (group) of class (C_0), then $\{\underline{S}_t\}$ with $\underline{S}_t = e^{\alpha t} \underline{T}_t$ is a semi-group (group) of class (C_0) for constant $\alpha \varepsilon (-\infty, \infty)$.

The group structure[23] provides the norm of $\underline{T}_t$, where $\{\underline{T}_t\}$ is a group of class (C_0) with an upper bound--there exist constants $M > 0$ and $\beta < \infty$ such that

$$||\underline{T}_t|| \leqq M\, e^{\beta |t|} \quad (-\infty < t < \infty). \qquad \text{(IV-7)}$$

If in (IV-7) $\beta = 0$, then $\{\underline{T}_t\}$ is an <u>equi-bounded group of class (C_0)</u>. And if in addition $M = 1$ in (IV-7), then $\{\underline{T}_t\}$ is a contraction group of class (C_0).

In the stability theory of semi-groups the contraction semi-groups of class (C_0) are very important. The nature of the groups, however, does not limit the stability properties to contraction groups of class (C_0). An extremely important theorem in proving necessary and sufficient conditions for asymptotic stability of groups was proven by Vogt:

Theorem IV-1. (Vogt). Let $\{\underline{T}_t\ ;\ t \in (-\infty,\infty)\} \subseteq L(X,X)$ be a group. Then there exist positive constants m and α such that

$$||\underline{T}_t\ \underline{x}|| \geqq M\ e^{-\alpha|t|}||\underline{x}|| \qquad (-\infty < t < \infty) \qquad \text{(IV-8)}$$

Proof: Since $\underline{T}_t \cdot \underline{T}_{-t} = \underline{T}_0 = \underline{I}$,

$$||\underline{x}|| = ||\underline{T}_{-t}\ \underline{T}_t\ \underline{x}|| \leq ||\underline{T}_{-t}||\,||\underline{T}_t\ \underline{x}||$$

or

$$||\underline{T}_t\ \underline{x}|| \geq ||\underline{T}_{-t}||^{-1}||\underline{x}||$$

From (IV-7):

$$||\underline{T}_{-t}||^{-1} \geqq M^{-1}\ e^{-\beta|t|}$$

and hence

$$||\underline{T}_t\ \underline{x}|| \geq M^{-1}\ e^{-\beta|t|}\ ||x||$$

which is the required inequality.

It should be noted that (IV-8) is a sufficient condition for the existence of the inverse $\underline{S}_t^{-1} = \underline{S}_{-t}$ for each $t \in (-\infty,\infty)$ with $\mathcal{D}(\underline{S}_{-t}) = X$.

B. Infinitesimal Generators of Semi-Groups and Groups of Class (C_0)

In the previous section, the semi-group or group system structure has been established. The next step is to derive the conditions for $\underline{A}$ so that the solutions to (IV-1) possess this structure. In the next chapter the Lyapunov stability definitions and theory is established for these semi-groups or groups.

Definition IV-3. Let $\{\underline{T}_t\ ;\ t \geqq 0\} \subseteq L(X,X)$ be an equi-continuous semi-group of class (C_0). The infinitesimal generator $\underline{A}$ of $\underline{T}_t$ is defined by

$$\underline{A}\ \underline{x} = \lim_{h\to 0^+}\ \{h^{-1}\ (\underline{T}_h - \underline{I})\underline{x}\} \qquad \text{(IV-9)}$$

whenever this limit exists.

$\underline{A}$ is a linear operator with domain

$$\mathcal{D}(\underline{A}) = \{\underline{x} \in X; \lim_{h\to 0^+} [h^{-1} (\underline{T}_h - \underline{I})\underline{x}] \text{ exists in } X\} \qquad \text{(IV-10)}$$

and the range of $\underline{A}$, $\mathcal{R}(\underline{A}) \subseteq X$. Since $\mathcal{D}(\underline{A})$ contains at least the vector $\underline{0}$, it is nonempty. In fact $\overline{\mathcal{D}(\underline{A})} = X$[23].

The following theorem gives the necessary and sufficient conditions for $\underline{A}$ to be an infinitesimal generator of a semi-group. The proof of the theorems in this section can be found in[23].

Theorem IV-2. Let $\underline{A}$ be a linear operator with $\mathcal{D}(\underline{A})$ dense in X and $\mathcal{R}(\underline{A})$ in X and let the resolvent $(\underline{I} - n^{-1}\underline{A})^{-1}$ exist in L(X,X). Then $\underline{A}$ is the infinitesimal generator of a uniquely determined equi-continuous semi-group if and only if there exists a positive constant c independent of n and m such that

$$||(\underline{I} - n^{-1}\underline{A})^{-m}|| \leqq c \quad (n=1,2,3, \ldots ; m=1,2,3,\ldots). \qquad \text{(IV-11)}$$

The semi-group generated by $\underline{A}$ in the above theorem is basically related to the spectral properties of $\underline{A}$ in that $(\lambda\underline{I} - \underline{A})^{-1}$ exists everywhere for $\mathrm{Re}(\lambda) > 0$ if $\underline{A}$ generates a semigroup statisfying $||\underline{T}_t|| \leqq M$.

Corollary IV-1. If in Theorem IV-2, (IV-11) is replaced by

$$||(\underline{I} - n^{-1}\underline{A})^{-1}|| \leqq 1 \qquad (n=1,2,3,\ldots) \qquad \text{(IV-12)}$$

then $\underline{A}$ generates a contraction semi-group.

The following theorem relates certain semi-groups and their corresponding infinitesimal generators.

Theorem IV-3. Let $\underline{A}$ be the infinitesimal generator of the semi-group $\{\underline{T}_t\}$ satisfying

i. $\underline{T}_t \cdot \underline{T}_s = \underline{T}_{t+s}$ $(t,s \geqq 0)$, $\underline{T}_0 = \underline{I}$

ii. $\lim_{t\to 0^+} ||\underline{T}_t\underline{x} - \underline{x}|| = 0 \;\forall\; \underline{x} \in X$

iii. $||\underline{T}_t|| \leqq M\, e^{\beta t}$ $(t \geqq 0)$ with $M > 0$ and $\beta < \infty$ and independent of t. Then $(\underline{A} - \beta\, \underline{I})$ is the infinitesimal generator of the equi-continuous semi-group $\underline{S}_t = e^{-\beta t}\, \underline{T}_t$ of class (C_0) and $(\lambda\, \underline{I} - \underline{A})^{-1}$ is everywhere defined for $Re[\lambda] > \beta$.

The above theorems IV-2 and IV-3 give rise to the following corollary:

Corollary IV-2. Let $\underline{A}$ be a closed linear operator with $\overline{\mathcal{D}(\underline{A})} = X$ and $\mathcal{R}(\underline{A}) \subseteqq X$ and let the resolvent $(\underline{I} - n^{-1}\, \underline{A})^{-1}$ exist in L(X,X) for integer n sufficiently large. Then $\underline{A}$ is the infinitesimal generator of a semi-group satisfying (i), (ii), and (iii) of Theorem IV-3 if and only if there exist constants, $M > 0$ and $\beta < \infty$ such that

$$||(\underline{I} - n^{-1}\underline{A})^{-m}|| \leqq M(1 - n^{-1}\beta)^{-m} \text{ for } m=1,2,3, \ldots \text{ and all large } n. \quad \text{(IV-13)}$$

In particular for those semi-groups $\underline{T}_t$ satisfying (i), (ii), and

$$||\underline{T}_t|| \leqq e^{\beta t} \text{ for all } t \geqq 0, \quad \text{(IV-14)}$$

(IV-13) can be replaced by

$$||(\underline{I} - n^{-1}\, \underline{A})^{-1}|| \leqq (1 - n^{-1}\beta)^{-1} \text{ for all large } n. \quad \text{(IV-15)}$$

Notice that β can be positive as well as negative in cases (IV-13) and (IV-15).

These results relate semi-groups and their infinitesimal generators. Similar results hold for groups and their infinitesimal generators. The following theorem summarizes these results:

Theorem IV-4. Let $\underline{A}$ be a linear operator with $\overline{\mathcal{D}(\underline{A})} = X$ and $\mathcal{R}(\underline{A})$ in X. Let the resolvent $(\underline{I} - n^{-1}\, \underline{A})^{-1}$ exist in X. Then $\underline{A}$ is the infinitesimal generator of a uniquely determined equi-continuous group $\{\underline{T}_t;\ t \in (-\infty, \infty)\}$ of class (C_0) if and only if there exists a positive constant $M > 0$ such that

$$||(\underline{I} - n^{-1}\underline{A})^{-m}|| \leqq M \; (m=1,2,3, \ldots \text{ and all large } |n|,\ n \lessgtr 0)$$

(IV-16)

and

$$||\underline{T}_t|| \leq M \quad (-\infty < t < \infty). \tag{IV-17}$$

If in addition $\underline{A}$ is closed and there exist positive constants $M > 0$, $\beta \geq 0$ such that

$$||(\underline{I} - n^{-1}\underline{A})^{-m}|| \leq M(1 - |n^{-1}|\beta)^{-m} \quad (m=1,2,3,\ldots \text{large}|n|,\ n \gtrless 0) \tag{IV-18}$$

then $\{\underline{T}_t;\ -\infty < t < \infty\}$ satisfies

$$||\underline{T}_t|| \leq M\, e^{\beta|t|} \quad (-\infty < t < \infty). \tag{IV-19}$$

If

$$||(\underline{I} - n^{-1}\underline{A})^{-1}|| \leq (1 - |n^{-1}|\ \beta)^{-1} \quad (\text{for large } |n|,\ n \gtrless 0) \tag{IV-20}$$

then $\{\underline{T}_t;\ -\infty < t < \infty\}$ satisfies

$$||\underline{T}_t|| \leq e^{\beta|t|} \quad (-\infty < t < \infty). \tag{IV-21}$$

The statement $\underline{A}$ is a closed linear operator in Corollary IV-2 and the last part of Theorem IV-4 is crucial. Further elaboration is possible for the case when $\underline{A}$, defined on a Hilbert space, is the infinitesimal generator of a contraction semi-group of class (C_0). These contraction semi-groups of class (C_0) are very important for stability investigations. Their infinitesimal generators are the earlier defined dissipative operators. The results of the following theorem and corollary are due to Phillips(24).

Theorem IV-5. (Phillips). Let $\underline{A}$ be a linear operator with domain $\mathcal{D}(\underline{A})$ and range $\mathcal{R}(\underline{A})$ both in the Hilbert space H and $\overline{\mathcal{D}(\underline{A})} = H$, then $\underline{A}$ generates a contraction semi-group of class (C_0) in H if and only if $\underline{A}$ is dissipative with respect to the inner product defined on H and $\mathcal{R}(\underline{I} - \underline{A}) = H$.

A consequence of this theorem is the following corollary:

Corollary IV-3. If $\underline{A}$ is a closed linear operator with $\mathcal{D}(\underline{A})$ and $\mathcal{R}(\underline{A})$ both in the Hilbert space H and $\overline{\mathcal{D}(\underline{A})} = H$, then $\underline{A}$ generates a contraction semi-group of class (C_0) in H if $\underline{A}$ and its adjoint $\underline{A}^*$, are both dissipative with respect to the inner product defined on H.

In the corollary it is again required that $\underline{A}$ is closed, a statement not made in the theorem.

The principal requirement for $\underline{A}$ to generate a contraction semi-group is that $\mathcal{D}(\underline{A})$ is dense in H and that $\underline{A}$ is maximally dissipative, i.e., $\underline{A}$ is not the proper restriction of any other dissipative operator. A necessary and sufficient condition is that $\mathcal{R}(\lambda\, \underline{I} - \underline{A}) = H$ for all $\lambda > 0$. In this case $\underline{A}$ is closed if and only if $\mathcal{R}(\lambda\, \underline{I} - \underline{A})$ is closed. However, if no conditions are imposed on $\mathcal{R}(\lambda\, \underline{I} - \underline{A})$ or equivalently on $\mathcal{R}(\underline{I} - \underline{A})$, then $\underline{A}$ must be closed.

After having developed the relationship between the semi-group or group and the infinitesimal generator, there remains one more step to identify systems having the semi-group property with the operator differential equation (IV-1). Let D_t denote the time derivative of $\underline{T}_t\, \underline{x}$ for $\underline{x} \in X$ and define this derivative by

$$D_t\, \underline{T}_t\, \underline{x} = \lim_{h \to 0^+} \left[h^{-1}\, (\underline{T}_{t+h} - \underline{T}_t)\, \underline{x}\right] \qquad \text{(IV-22)}$$

for $\underline{x} \in X$ if the limit exists.

From Yosida[23]

<u>Theorem IV-6</u>. If $\underline{x} \in \mathcal{D}(\underline{A}) \subseteq X$, then $\underline{x} \in \mathcal{D}(D_t \underline{T}_t)$ and

$$D_t\, \underline{T}_t\, \underline{x} = \underline{A}\, \underline{T}_t\, \underline{x} = \underline{T}_t\, \underline{A}\, \underline{x} \quad \text{for } t \geq 0. \qquad \text{(IV-23)}$$

And in particular, $\underline{A}$ is commutative with $\underline{T}_t$ for $\underline{x} \in \mathcal{D}(\underline{A})$. Thus if $\underline{x} \in \mathcal{D}(\underline{A})$, then $\underline{T}_t\, \underline{x} \in \mathcal{D}(\underline{A})$ for $t \geq 0$. In (IV-23) of the above theorem, one easily recognizes the operator differential equation:

$$\frac{d\, \underline{x}}{dt} = \underline{A}\, \underline{x} \qquad (\underline{x} \in \mathcal{D}(\underline{A}) \subseteq X). \qquad \text{(IV-24)}$$

Thus if in accordance with the above theorems proper restrictions are placed on $\underline{A}$, then the solutions to (IV-24) are the semi-group or group trajectories given by the semi-group or group that is uniquely generated by $\underline{A}$. The properties of the solutions can be studied by investigating the properties of this semi-group, i.e., of the infinitesimal generator $\underline{A}$ as given by (IV-24).

V. LYAPUNOV STABILITY THEORY FOR SEMI-GROUPS AND GROUPS

A. Definitions

Once the semi-group or group structure is established for the general operator differential equations

$$\frac{d\underline{x}}{dt} = \underline{A}\,\underline{x} \quad (\underline{x}\ \varepsilon\,\mathcal{D}(\underline{A}) \subseteq X) \tag{V-1}$$

then the solutions are semi-group or group trajectories. Thus the solution starting at t=0 from $\underline{x} = \underline{x}_0\ \varepsilon\mathcal{D}(\underline{A})$ is given by

$$\underline{x}(t;\underline{x}_0) = \underline{T}_t\,\underline{x}_0 \qquad t \geqq 0 \tag{V-2}$$

with $\underline{x}(0;\underline{x}_0) = \underline{x}_0$, where $\underline{A}$ is the infinitesimal generator of the semi-group $\{\underline{T}_t\}$. If $\underline{A}$ is the infinitesimal generator of a group $\{\underline{T}_t\}$, then (V-2) holds for $-\infty < t < +\infty$. Note that by Theorem IV-6, $\underline{x}(t;\underline{x}_0)\varepsilon\,\mathcal{D}(\underline{A})$ $(t \geqq 0)$ if $\underline{x}_0\varepsilon\mathcal{D}(\underline{A})$. By the linearity of the semi-group, any trajectory can be referenced to the origin $\underline{x}=\underline{0}$. Thus the stability of any solution can be determined by studying the stability of the solution $\underline{x}(t;\underline{x}_0) = \underline{0}$, the origin or null solution. It is now possible to give the definitions of stability in terms of the semi-group $\underline{T}_t$ generated by $\underline{A}$ of (V-1).

Definition V-1. The origin of (V-1) is stable in the sense of Lyapunov (with respect to initial perturbations) if and only if, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$||\underline{x}_0|| < \delta \tag{V-3}$$

implies that

$$||\underline{T}_t\,\underline{x}_0|| < \varepsilon \qquad (t \geqq 0;\ \forall\ \underline{x}_0\ \varepsilon\ X). \tag{V-4}$$

Definition V-2. The origin of (V-1) is asymptotically stable in the sense of Lyapunov (with respect to initial perturbations) if and only if

i. it is stable

ii. $$\lim_{t\to\infty} ||\underline{T}_t \, \underline{x}_0|| = 0. \quad (\forall \, \underline{x}_0 \, \varepsilon X). \tag{V-5}$$

The exponential nature of the semi-group structure usually gives rise to a stronger form of asymptotic stability, namely exponential asymptotic stability as defined by:

Definition V-3. The origin of (V-1) is exponentially asymptotically stable in the sense of Lyapunov (with respect to initial perturbations) if and only if

i. it is asymptotically stable

ii. there exist positive constants M and β such that

$$||\underline{T}_t \, \underline{x}_0|| \leqq M \, e^{-\beta t} \, ||\underline{x}_0|| \quad (\forall \, \underline{x}_0 \, \varepsilon \, X). \tag{V-6}$$

From Definition V-3 it is clear that $||\underline{T}_t|| \leqq M \, e^{-\beta t}$. If $\{\underline{T}_t\}$ is a group, then from Theorem IV-1 there follows for $t \geqq 0$, $||\underline{T}_t|| \geqq m \, e^{-\alpha t}$. A group $\{\underline{T}_t\}$ with the property that there exist four positive constants, $M \geqq 1 \geqq m > 0, \alpha \geqq \beta > 0$ such that

$$m \, e^{-\alpha t} ||\underline{x}|| \leqq ||\underline{T}_t \, \underline{x}|| \leqq M \, e^{-\beta t} \, ||\underline{x}|| \quad (t \geqq 0, \, \underline{x} \, \varepsilon \, X) \tag{V-7}$$

is called a group of exponential type.

The following theorems are direct consequences of the above definitions and those in the preceding chapter:

Theorem V-1. A sufficient condition for the stability of the null solution of (V-1) is that the semi-group $\{\underline{T}_t\}$ be equi-bounded.

Theorem V-2. A sufficient condition for the exponential asymptotic stability of the null solution of (V-1) is that there exist positive constants M and β such that

$$||\underline{T}_t|| \leqq M\, e^{-\beta t}. \tag{V-8}$$

B. Sufficient Conditions for Stability and Asymptotic Stability of Semi-Groups

In the preceding section the stability properties are defined in terms of the semi-group properties. Thus, when this semi-group is generated by an operator, it is in terms of the solutions to the operator differential equation. However, rather than first solving the equation, one would like to base the stability properties of the system directly on the properties of the operator, i.e., to determine conditions for the operator so that the solutions exist and at the same time are stable. These conditions are clearly spelled out in the theorems in Chapter IV, Section B.

It is intuitive that stability of a null solution requires $\underline{A}$ to be the infinitesimal generator of an equibounded semi-group. However, it is not so easy to relate asymptotic or exponential asymptotic stability directly to such a basic property. However, once the contraction property is established (IV-14) and (IV-15) of Corollary IV-2 seem to provide the answer.

In Theorem IV-5 the principal conditions on $\underline{A}$ to generate a contraction semi-group are that $\underline{A}$ is a dissipative operator with respect to an inner product $<\cdot,\cdot>_1$ and $\mathcal{R}(\underline{I} - \underline{A}) = H_1$ and consequently the null solution is stable.

The dissipativity is defined with respect to the particular inner product of the space, but the semi-group property is invariant under equivalent norming. In general, a semi-group $\{\underline{T}_t\}$ is stable if it is equi-bounded, i.e., $||\underline{T}_t||_1 \leqq M$. The invariance under equivalent norming does not necessarily mean that if $||\underline{T}_t||_1 \leqq M \Leftrightarrow ||\underline{T}_t||_2 \leqq 1$ then $||\cdot||_2$ corresponds to an inner product $<\cdot,\cdot>_2$. In other words, $||\cdot||_2$ may define a Banach space rather than a Hilbert space.

The invariance of stability under equivalent norming suggests that if $\underline{A}$ is dissipative with respect to any inner product equivalent to the inner product of the space, then the dissipative property of $\underline{A}$ with respect to this inner product is sufficient for stability and generation of a contraction semi-group. For this reason the dissipativity of $\underline{A}$ is extended as follows:

Definition V-4. Let H_1 be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_1$. $\underline{A}$ is dissipative in the extended sense if and only if there exists a self-adjoint, positive definite, bounded linear operator $\underline{S} \in L(H,H)$ such that

$$\langle \underline{x}, \underline{S}\,\underline{A}\,\underline{x} \rangle_1 \leqq 0 \quad \underline{x} \in \mathcal{D}(\underline{A}) \tag{V-9}$$

and $\underline{A}$ is strictly dissipative in the extended sense if and only if there exists a $\beta > 0$ such that

$$\langle \underline{x}, \underline{S}\,\underline{A}\,\underline{x} \rangle_1 \leqq -\beta \langle \underline{x}, \underline{S}\,\underline{x} \rangle_1 \quad \underline{x} \in \mathcal{D}(\underline{A}) \tag{V-10}$$

Since under the conditions on $\underline{S}$:

$$\langle \underline{x}, \underline{y} \rangle_2 = \langle \underline{x}, \underline{S}\,\underline{y} \rangle_1 \quad \underline{x}, \underline{y} \in H$$

one could also call $\underline{A}$ dissipative in the extended sense if there exists an equivalent inner product with respect to which $\underline{A}$ is dissipative. The following theorem follows directly from Theorem IV-5.

Theorem V-3. Let $\underline{A}$ be a linear operator with $\mathcal{D}(\underline{A}) = H$ and $\mathcal{R}(\underline{A}) \subseteq H$ and $\mathcal{R}(\underline{I} - \underline{A}) = H$. Then $\underline{A}$ generates a contraction semi-group if and only if $\underline{A}$ is dissipative in the extended sense.

Proof: Since $\underline{A}$ is dissipative in the extended sense there exists an inner product with respect to which $\underline{A}$ is dissipative. The "if" part follows from Theorem IV-5 by using this inner product as the inner product for the space.

If $\underline{A}$ generates a contraction semi-group according to $||\cdot||^2 = \langle \cdot, \cdot \rangle$, then $\underline{A}$ is dissipative with respect to $\langle \cdot, \cdot \rangle$ and hence is dissipative in the extended sense.

Remark: Theorem V-3 includes Theorem IV-5, in the sense that $\underline{A}$ is not dissipative with respect to the original inner product of the space. In this case $\{\underline{T}_t;\ t \geq 0\}$ is at least equi-bounded with respect to the norm induced by the original inner product.

In order to derive some principal results on the stability and asymptotic stability of certain classes of semi-groups, Lyapunov functionals will be introduced.

A Lyapunov functional on a real Hilbert space H_1 is defined through the symmetric, bilinear form,

$$V(\underline{x},\underline{y}) = \langle \underline{x}, \underline{S}\,\underline{y}\rangle_1 = \langle \underline{y}, \underline{S}\,\underline{x}\rangle_1 \quad \underline{x}, \underline{y} \in H \tag{V-11}$$

where $\underline{S}$ is a self-adjoint, bounded positive definite, linear transformation, $\underline{S} \in L(H,H)$. Define the Lyapunov Functional by

$$v(\underline{x}) = V(\underline{x},\underline{x}) \qquad x \in H. \tag{V-12}$$

The time derivative of $v(\underline{x})$, denoted by $\dot{v}(\underline{x})$ along solutions to (V-1) with $\underline{A}$ generating a semi-group $\{\underline{T}_t;t \geq 0\} \subseteq L(H,H)$ is given by:

$$\dot{v}(\underline{x}) = \lim_{t\to 0} \frac{1}{t}\ (V(\underline{T}_t\,\underline{x}, \underline{T}_t\,\underline{x}) - V(\underline{x},\underline{x})) \tag{V-13}$$

whenever this limit exists.

But with

$$V(\underline{T}_t\,\underline{x} + \underline{x}, \underline{T}_t\,\underline{x} - \underline{x}) = V(\underline{T}_t\,\underline{x}, \underline{T}_t\,\underline{x}) + V(\underline{x}, \underline{T}_t\,\underline{x}) - V(\underline{T}_t\,\underline{x},\underline{x}) - V(\underline{x},\underline{x})$$

and since $V(\underline{x},\underline{y})$ is symmetric it follows that:

$$\dot{v}(\underline{x}) = \lim_{t\to 0} \frac{1}{t} V((\underline{T}_t + \underline{I})\,\underline{x}, (\underline{T}_t - \underline{I})\,\underline{x})$$

$$= 2\,V(\underline{x}, \underline{A}\,\underline{x}) \qquad (\underline{x} \in \mathcal{D}(\underline{A})). \tag{V-14}$$

This leads to an important result:

Theorem V-4. Let $\underline{A}$ be a linear operator with $\overline{\mathcal{D}(\underline{A})} = H$, $\mathcal{R}(\underline{A}) \subseteq H$, and $\mathcal{R}(\underline{I} - \underline{A}) = H$. Then $\underline{A}$ is an infinitesimal generator of a contraction semi-group if and only if there exists a Lyapunov Functional $v(\underline{x})$ such that

$$\dot{v}(\underline{x}) = 2V(\underline{x}, \underline{A}\,\underline{x}) \leqq 0 \quad \underline{x} \in \mathcal{D}(\underline{A}).$$

Corollary V-4.1. Under the conditions of Theorem V-4, the null solution of (V-1) is stable.

Proof: The "only if" part of the theorem follows from Theorem V-3 by taking $v(\underline{x})$ as,

$$v(\underline{x}) = \langle \underline{x}, \underline{x} \rangle.$$

If there exists a $v(\underline{x})$ given by $V(\underline{x},\underline{y})$, then let $\langle \underline{x},\underline{y} \rangle_2 = V(\underline{x},\underline{y})$, with $\langle \underline{x}, \underline{A}\,\underline{x} \rangle_2 \leqq 0$. Thus by Theorem V-3, since $\overline{\mathcal{D}(\underline{A})} = H$ and $\mathcal{R}(\underline{I} - \underline{A}) = H$, $\underline{A}$ generates a contraction semi-group $\{\underline{T}_t;\ t \geqq 0\}$ with

$$||\underline{T}_t||_2 \leqq 1 \quad t \geqq 0.$$

This result implies Corollary V-4.1.

The statement of the theorem indicates that the conditions for existence and stability of the solutions to a general operator differential equation of the form (V-1) are much more restrictive than when (V-1) represents a finite dimensional system of ordinary differential equations. The free interchange of equivalent inner products can facilitate the investigation of these requirements.

The following theorem points out its importance for the case of asymptotic stability.

Theorem V-5. Let $\underline{A}$ be a linear operator with $\overline{\mathcal{D}(\underline{A})} = H$, $\mathcal{R}(\underline{A}) \subseteq H$, $\mathcal{R}(\underline{I} - \underline{A}) = H$. Then the null solution of (V-1) is asymptotically stable if there exists Lyapunov Functional $v(\underline{x})$ such that

$$\dot{v}(\underline{x}) = 2V(\underline{x}, \underline{A}\,\underline{x}) \leqq -\gamma\, ||\underline{x}||_1^2 \quad x \in \mathcal{D}(\underline{A}).$$

Proof: Since all the conditions of Theorem V-4 are satisfied it follows that $\underline{A}$ generates a contraction semi-group. To show that the null solution is asymptotically stable notice that from

$$\dot{V}(\underline{x}) = 2V(\underline{x}, \underline{A}\,\underline{x}) \leqq -\gamma\ ||\underline{x}||_1^2$$

certainly follows

$$V(\underline{x}, \underline{A}\,\underline{x}) = <\underline{x}, \underline{A}\,\underline{x}>_2 \leqq -\alpha\ ||\underline{x}||_2^2\ .$$

Next it must be shown that for $n \geqq N > 0$, the inverse of $(I - n^{-1}A)$ exists such that

$$||(\underline{I} - n^{-1}\underline{A})^{-1}||_2 \leqq (1 - n^{-1}\beta)^{-1}$$

for some β. From the relation

$$<(\underline{I} - n^{-1}\underline{A})\,\underline{x},\underline{x}>_2 = <\underline{x},\underline{x}>_2 - n^{-1} <\underline{A}\,\underline{x},\underline{x}>_2$$

it follows from Schwarz's inequality that

$$||(\underline{I} - n^{-1}\underline{A})\underline{x}||_2||\underline{x}||_2 \geqq |<(\underline{I} - n^{-1}\underline{A})\underline{x},\underline{x}>_2| =$$

$$= <\underline{x},\underline{x}>_2 \left| 1 - n^{-1} \frac{<\underline{A}\,\underline{x},\underline{x}>_2}{<\underline{x},\underline{x}>_2} \right|$$

and therefore that

$$||(\underline{I} - n^{-1}\underline{A})^{-1}||_2 \leqq (1 - n^{-1}(-\alpha))^{-1} \qquad n \geqq \alpha.$$

It follows that

$$||\underline{T}_t||_2 \leqq e^{-\alpha t}.$$

This is a sufficient condition for the asymptotic stability of the null solution.

Remark. The above theorems can similarly be stated with the condition of $\underline{A}$ being closed and with the subsequent modifications following the theorems of Chapter IV.

Note also that the Lyapunov Functional gives only sufficient conditions for stability or asymptotic stability respectively. This is a distinct difference between the stability properties of systems having the semi-group and group properties respectively, a fact clearly demonstrated in the following section. It is foreseen that a slight conceptual change in the selection of the Lyapunov Functional may alleviate this difficulty.

C. Necessary and Sufficient Conditions for the Exponential Asymptotic Stability of Groups

In the preceding section the sufficient condition for the asymptotic stability of semi-groups in terms of its infinitesimal generator was established. This applies also to groups. However, the group property is the fundamental property of dynamical systems for which a Lyapunov stability theory has been developed which includes both sufficient and necessary conditions for stability or asymptotic stability. Thus, it is not surprising that in the case of groups the approach developed in the previous section can be extended to include the necessity. Again the Lyapunov Functional not only gives necessary and sufficient conditions for stability or asymptotic stability but guarantees also the existence of the solutions.

Theorem V-6. Let $\underline{A}$ be a linear operator with $\overline{\mathcal{D}(\underline{A})} = H$ with inner product $<\cdot,\cdot>_1$ and such that $\mathcal{R}(\lambda \underline{I} - \underline{A}) = H$ for real λ and $|\lambda|$ sufficiently large. Then $\underline{A}$ is the infinitesimal generator of a group of exponential type if and only if there exists a Lyapunov Functional $v(\underline{x}) = V(\underline{x},\underline{x})$ with $V(\underline{x},\underline{y}) = <\underline{x}, \underline{S}\,\underline{y}>_1 = <\underline{S}\,\underline{x}, \underline{y}>_1$ where $\underline{S}$ is a symmetric or self-adjoint bounded, positive definite operator, $\underline{S} \in L(H,H)$ and such that for some constants α and β, $\alpha > \beta > 0$

$$- 2\,\alpha V(\underline{x},\underline{x}) \leqq \dot{v}(\underline{x}) = 2V(\underline{x}, \underline{A}\,\underline{x}) \leqq -2\,\beta\, V(\underline{x},\underline{x}). \quad (x \varepsilon \mathcal{D}(\underline{A})) \qquad \text{(V-15)}$$

Proof: If there exists a $V(\underline{x},\underline{y})$ with the above properties, then let $<\cdot,\cdot>_2 = V(\cdot,\cdot)$. To show that $\underline{A}$ generates a group there follows:

$$< (\underline{I} - n^{-1}\underline{A})\,\underline{x},\underline{x}>_2 \; = \; < \underline{x},\underline{x}>_2 - n^{-1} < \underline{A}\,\underline{x},\, \underline{x}>_2 \quad .$$

Then with Schwarz's inequality:

$$||(\underline{I} - n^{-1}\underline{A})\underline{x}||_2 \geqq ||\underline{x}||_2 \left|1 - n^{-1}\frac{< \underline{A}\,\underline{x},\underline{x}>_2}{< \underline{x},\underline{x}>_2}\right| .$$

For $\qquad n \geqq N_\beta > 0,$

$$||\underline{I} - n^{-1}\,\underline{A}||_2 \geqq 1 + |n^{-1}|\beta \qquad \text{(V-16)}$$

and for $\qquad n \leqq - N_\alpha < 0$

$$||\underline{I} - n^{-1}\,\underline{A}||_2 \geqq 1 - |n^{-1}|\,\alpha.$$

Thus for $\qquad |n| \geqq \max\,(N_\alpha, N_\beta) \qquad n \lessgtr 0$

$$||\underline{I} - n^{-1}\,\underline{A}||_2 \geqq 1 - |n^{-1}|\,\alpha > 0.$$

Therefore $(\underline{I} - n^{-1}\,\underline{A})^{-1}$ exists and

$$||(\underline{I} - n^{-1}\,\underline{A})^{-1}||_2 \leqq (1 - |n^{-1}|\alpha)^{-1}$$

and thus by Theorem IV-4 $\underline{A}$ generates a group $\{\underline{T}_t;\ t\ \varepsilon\ (-\infty, \infty)\}$ which satisfies

$$||\underline{T}_t||_2 \leqq e^{\alpha|t|} \qquad (-\infty < t < \infty).$$

From inequality (V-16) it follows by Corollary IV-2 that $\underline{A}$ generates a semigroup $\{\underline{T}_t;\ t \geqq 0\}$ with

$$||\underline{T}_t||_2 \leqq e^{-\beta t} \qquad t\ \varepsilon\ [0, \infty).$$

Since $||\cdot||_1$ and $||\cdot||_2$ are equivalent, this completes the proof of sufficiency. The only if part will be demonstrated by construction of the required Lyapunov Functional. If $\underline{A}$ generates an exponentially asymptotically stable group, then it follows from Theorem IV-1 that the most general group $\{\underline{T}_t;\ t\varepsilon(-\infty,\infty)\}$ satisfies

$$m\ e^{-\alpha t}\,||\underline{x}||_1 \leq ||\underline{T}_t\ \underline{x}||_1 \leq M\ e^{-\beta t}\ ||\underline{x}||_1 \qquad t\varepsilon[0,\infty)$$

where $\infty > M \geqq 1 \geqq m > 0$, $\infty > \alpha > \beta > 0$ and $||\underline{T}_t||_1 \leqq M\ e^{\alpha|t|}$ $(-\infty < t < \infty)$.

Next take

$$V(n;\ \underline{x},\underline{y}) = \int_0^n <\underline{T}_t\ \underline{x},\ \underline{T}_t\ \underline{y}>_1\ dt \qquad \text{(V-17)}$$

with $n > 0$. Since $<\underline{T}_t\underline{x},\ \underline{T}_t\ \underline{y}>_1$ is for each t a numerical value, which is everywhere defined and continuous on the compact interval [0,n] (V-17) can be interpreted as a Riemann integral. For each fixed n, $V(n;\ \underline{x},\underline{y})$ is a symmetric bilinear form, which satisfies

$$|V(n;\ \underline{x},\underline{y})| \leqq ||\underline{x}||_1\ ||\underline{y}||_1\ M^2 \int_0^n e^{-2\beta t}\ dt$$

$$= ||\underline{x}||_1\ ||\underline{y}||_1\ \frac{M^2}{2\beta}(1 - e^{-2\beta n}). \qquad \text{(V-18)}$$

Next let $V(\underline{x},\underline{y}) = \lim\limits_{n\to\infty} V(n;\ \underline{x},\underline{y})$, then from (V-18) it follows that $V(\underline{x},\underline{y})$ exists and is well defined, moreover

$$|V(\underline{x},\underline{y})| \leqq \frac{M^2}{2\beta}\,||\underline{x}||_1\ ||\underline{y}||_1$$

and $$V(\underline{x},\underline{x}) \geqq \frac{m^2}{2\alpha}\ ||x||_1^2\ .$$

The symmetric bilinear functional $V(\underline{x},\underline{y})$ satisfies all the conditions of the Lax-Milgram Theorem, thus there exists a uniquely determined, bounded, symmetric, positive definite linear operator $\underline{S}$ with

$$V(\underline{x},\underline{y}) = <\underline{x},\ \underline{S}\ \underline{y}>_1 = <\underline{S}\ \underline{x},\ \underline{y}>_1 \quad \underline{x},\ \underline{y}\ \varepsilon H$$

with

$$||\underline{S}||_1 \leqq \frac{M^2}{2\beta}\ .$$

Next let $v(\underline{x}) = V(\underline{x},\underline{x})$. Then there remains to be proven that $\dot{v}(\underline{x}) = V(\underline{x},\ \underline{A}\ \underline{x})$ satisfies the conditions of Theorem V-6. But

$$V(\underline{T}_t\,\underline{x},\ \underline{T}_t\,\underline{x}) - V(\underline{x},\underline{x})$$

$$= \lim_{n\to\infty} \left[\int_0^n < \underline{T}_\tau \underline{T}_t\,\underline{x},\ \underline{T}_\tau\,\underline{T}_t\,\underline{x}>_1 d\tau - \int_0^n < \underline{T}_\tau \underline{x},\ \underline{T}_\tau\,\underline{x}>_1 \right] d\tau$$

$$= \lim_{n\to\infty} \left[\int_t^{n+t} < \underline{T}_s\,\underline{x},\ \underline{T}_s\,\underline{x}>_1\ ds - \int_0^n < \underline{T}_s\,\underline{x},\ \underline{T}_s\,\underline{x}>_1\ ds \right]$$

$$= \lim_{n\to\infty} \left[\int_n^{n+t} < \underline{T}_s\,\underline{x},\ \underline{T}_s\,\underline{x}>_1\ ds \right] - \int_0^t < \underline{T}_s \underline{x},\ \underline{T}_s\,\underline{x}>_1\ ds.$$

Therefore

$$2V(\underline{x},\underline{A}\,\underline{x}) = \lim_{t\to 0} \frac{1}{t} \left[V(\underline{T}_t\,\underline{x},\ \underline{T}_t\,\underline{x}) - V(\underline{x},\underline{x}) \right] =$$

$$= - \lim_{t\to 0} \frac{1}{t} \int_0^t < \underline{T}_s\,\underline{x},\ \underline{T}_s\,\underline{x}>_1\ ds$$

$$= - < \underline{x},\underline{x}>_1 \qquad (\underline{x} \in \mathcal{D}(\underline{A}))$$

But since $<\cdot,\cdot>_1$ is equivalent to $<\cdot,\cdot>_2 = V(\cdot,\cdot)$, it follows that there exists $\alpha > \beta > 0$ such that

$$- \alpha V(\underline{x},\underline{x}) \leqq V(\underline{x},\ \underline{A}\,\underline{x}) \leqq - \beta\, V(\underline{x},\underline{x}).$$

This completes the proof of the theorem.

The interesting aspects of this theorem become clear in Section VIII, where certain wave equations are studied. There, the difference between the stability properties for the semi-group and group structures are demonstrated by taking the same simple operator studied in Section VII in a diffusion equation setting, and in Section VIII in a wave equation setting. In the next section, the formal partial differential operator is synthesized to give the appropriate representation (V-1).

VI. FORMULATION OF FORMAL PARTIAL DIFFERENTIAL OPERATORS AS UNBOUNDED OPERATORS AND THE SELECTION OF NORMS

In this chapter an unbounded operator $\underline{A}$ is obtained from a class of formal partial differential operators. This means that $\mathcal{D}(\underline{A})$ and $\mathcal{R}(\underline{A})$ are determined along with the transformation $\underline{A}\ \underline{x} = \underline{y}$ for certain $\underline{x}$.

This is carried out by means of a synthesis process. First the linear space in which the operator is defined is completed by introducing generalized derivatives. Some basic definitions and lemmas from the theory of distributions are given. Sobolev spaces are introduced; these spaces are significant for the determination of the domain and range of the unbounded operator. The Sobolev Imbedding Theorem is stated. The chapter concludes with a discussion of some aspects of the selection of norms for the stability problem.

A. Derivation of the Differential Operator from the Formal Partial Differential Operator

The derivation of the formal partial differential operator in the proper space setting is adapted from Dunford and Schwartz[29]. The real case is considered only; the results can be extended to the complex case.

1. Notations

The symbol J will denote an <u>index</u>, i.e., a k-tuple $J = (j_1,\ldots,j_k)$ of integers, $|J| = k$, $\min J = \min\limits_{1 \leq i \leq k} j_i$, $\max J = \max\limits_{1 \leq i \leq k} j_i$. The case J is vacuous is denoted by $J = 0$. The symbol R^n is the real Euclidean n-space. An index J will be said to be an index for R^n if $\min J \geq 1$ and $\max J \leq n$. If $\underline{x} \in R^n$, so that $\underline{x} = (x_1, x_2, \ldots, x_n)$ and J is an index for R^n, so that $J = (j_1, \ldots, j_k)$, $k = |J|$, then $\underline{x}^J$ will denote the expression $x_{j1}\, x_{j2} \ldots x_{jk}$.

The operations $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial s}$ of partial differentiation are sometimes written as ∂_{x_j} or ∂_j and ∂_s respectively. If J is an index for R^n and $|J| = k$, then the higher partial derivative

$$\frac{\partial^k}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_k}}$$

will be called a partial derivative of order $k = |J|$ and will be written ∂^J. If $|J| = 0$ the operator ∂^J is defined as the identity operator.

2. Formal Partial Differential Operator

If m is a positive integer, an expression

$$\tau = \sum_{|J| \leqq m} a_J(\underline{x}) \, \partial^J$$

where the real coefficients a_J are infinitely differentiable functions in an open set $\Omega \subseteqq R^n$, will be called a <u>formal partial differential operator defined in Ω</u> and m will be called the <u>order</u> of τ.

$$\tau^*(\cdot) = \sum_{|J| \leqq m} (-1)^J \partial^J [a_J(\underline{x}) \, (\cdot)]$$

is called the <u>formal adjoint</u> of τ. In particular, if $\tau = \tau^*$ then τ is called <u>formally self-adjoint</u>. If

$$\hat{\tau} = \sum_{|J| \leqq \hat{m}} \hat{a}_J(\underline{x}) \, \partial^J$$

is another formal partial differential operator defined in Ω, then for functions f which are infinitely often differentiable in Ω, $\hat{\tau}\,(\tau f)$ is also defined in Ω. This is written

$$\hat{\tau}\,\tau = \hat{\tilde{\tau}} = \sum_{|\tilde{J}| \leqq \hat{m}+m} b_{\tilde{J}}(\underline{x}) \, \partial^{\tilde{J}}.$$

Similarly the sum of τ and $\hat{\tau}$ is defined by

$$\tilde{\tau} = \tau + \hat{\tau} = \sum_{|J| \leqq \max(m,\hat{m})} (a_J(\underline{x}) + \hat{a}_J(\underline{x}))\, \partial^J .$$

Example VI-1. Let $\underline{x} = (x_1,x_2,x_3)$, then a particular τ can have the following form:

$$\tau = \sum_{|J| \leqq 4} a_J(\underline{x})\, \partial^J = a_4(\underline{x})\, \partial^4 + a_3(\underline{x})\, \partial^3 + a_2(\underline{x})\, \partial^2 + a_1(\underline{x})\, \partial + a_0(\underline{x})\, \partial^0 =$$

$$= a_{0,4,0}(\underline{x}) \frac{\partial^4}{\partial x_2^4} - a_{2,1,1}(\underline{x}) \frac{\partial^4}{\partial x_1^2 \partial x_2\, \partial x_3} + a_{0,2,1}(\underline{x}) \frac{\partial^3}{\partial x_2^2\, \partial x_3} +$$

$$+ a_{2,0,0}(\underline{x}) \frac{\partial^2}{\partial x_1^2} - a_{1,1,0}(\underline{x}) \frac{\partial^2}{\partial x_1\, \partial x_2} + a_0(\underline{x}).$$

Notice that the elements $a_J(\underline{x})$, $0 \leqq |J| \leqq 4$ can be zero, as is $a_1(\underline{x})$. In most practical applications, some components of $a_J(\underline{x})$, $0 \leqq |J| \leqq 4$, will be zero.

$$\tau^*(\cdot) = \sum_{|J| \leqq 4} (-1)^J\, \partial^J\, [a_J(\underline{x})(\cdot)] = \frac{\partial^4}{\partial x_2^4} [a_{0,4,0}(\underline{x})(\cdot)] -$$

$$- \frac{\partial^4}{\partial x_1^2\, \partial x_2\, \partial x_3} [a_{2,1,1}(\underline{x})(\cdot)] - \frac{\partial^3}{\partial x_2^2\, \partial x_3} [a_{0,2,1}(\underline{x})(\cdot)] +$$

$$+ \frac{\partial^2}{\partial x_1^2} [a_{2,0,0}(\underline{x})(\cdot)] - \frac{\partial^2}{\partial x_1\, \partial x_2} [a_{1,1,0}(\underline{x})(\cdot)] + a_0(\underline{x}).$$

3. Function Spaces $C^k(\Omega)$ and $C_0^k(\Omega)$

Let Ω be an open set in R^n and $\bar{\Omega}$ its closure. Then the set of scalar functions f defined on Ω with all partial derivatives of order not more than k existing and continuous is denoted by $C^k(\Omega)$. The set $C_0^k(\Omega)$ consists of those functions in $C^k(\Omega)$ which vanish outside a compact subset $\Omega' = \bar{\Omega}'$, $\bar{\Omega}'$ is a proper subset of Ω. Thus $C_0^k(\Omega) \subset C^k(\Omega)$. The set $C^k(\bar{\Omega})$ consists of all functions defined on $\bar{\Omega}$ having all partial derivatives of orders up to k inclusive at each point of Ω and such that each partial derivative has a continuous extension to $\bar{\Omega}$. If this is the case, $\partial^J f(\underline{x})$ is defined for $\underline{x} \in \bar{\Omega}$ and $|J| \leq k$ as the extension by continuity of $\partial^J f(\underline{x})$ from Ω to $\bar{\Omega}$. We accept $C_0^k(\bar{\Omega}) = C_0^k(\Omega)$; if $f \in C_0^k(\Omega) = C_0^k(\bar{\Omega})$, then $\partial^J f = 0$ on $\bar{\Omega} - \Omega$.

The spaces $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ are especially important.

4. Norm in $C^k(\Omega)$, $C^k(\bar{\Omega})$

$C^k(\Omega)$ is made into a F-space (a space which does not satisfy $||\alpha\underline{x}|| = |\alpha| \; ||\underline{x}||$ of the previously defined norm properties; $||\cdot||_F$ so defined is called a F-norm) as follows. Let K_m be an increasing sequence of compact subsets of Ω or $\bar{\Omega}$. Suppose that K_m is such that any compact subset of Ω belongs to one of the sets K_m. Then for a function f in one of the spaces $C^k(\Omega)$, $C^k(\bar{\Omega})$, place

$$\mu(f; J,m) = \sup_{\underline{x} \in K_m} |\partial^J f(\underline{x})|$$

and define the F-norm of f by

$$||f||_{C^k} = \sum_{m=1}^{\infty} \sum_{j=0}^{k} \sum_{|J|=j} \frac{1}{2^m 2^j j!} \frac{\mu(f;J,m)}{1 + \mu(f;J,m)}. \quad \text{(VI-1)}$$

This norm makes the space into a complete F-space. If $k < \infty$ and $\bar{\Omega}$ is compact, but not otherwise, the space $C^k(\bar{\Omega})$ is a Banach space under a norm equivalent to the F-norm such as $||\cdot||_{C^k(\bar{\Omega})} = \sup_{|J| \leq k} |\partial^J \cdot|$. It is in the sense of these norms that reference is made to the topology of $C^k(\bar{\Omega})$.

5. Distributions

It is essential to apply partial differential operators to complete spaces, i.e., to spaces with elements which all possess the necessary differentiability requirements in order to carry out all required mathematical operations properly. Suppose, that the operator

$$(\tau_0 f)\ (\underline{x}) = \frac{\partial^2 f}{\partial x_1 \partial x_2}\ (\underline{x}) \qquad \underline{x} \in R^2$$

defined for each function f in $C_0^\infty\ (R^2)$ is considered. This operator τ_0 is densely defined in $L^2(R^2)$, the space of all functions f such that $|f(\underline{x})|^2$ is Lebesgue integrable over R^n. But τ_0 is not closed. Let τ be its closure, then $\mathcal{D}(\tau)$ contains nondifferentiable functions. Which non-differentiable functions? One might expect the answer to be those (non-differentiable) functions f such that $\partial_{x_1} \partial_{x_2}$ f belongs to $L^2(R^2)$. Thus one should be able to define $\partial_{x_1} \partial_{x_2}$ for every function, differentiable or not, and irrespective of whether $\partial_{x_1} \partial_{x_2}$ f belongs to $L^2(R^2)$ or not. Such a "derivative" can no longer be an element of any space of functions, but can only be a "function" in some generalized sense. Laurent Schwartz has developed a theory of such "generalized functions" in his theory of distributions.

Definition VI-1. i. Let Ω be an open set in R^n. Let $\{\phi_n\}$ be a sequence of functions in $C_0^\infty(\Omega)$ and let ϕ be in $C_0^\infty(\Omega)$. Then if there is a compact subset K of Ω such that all functions ϕ_n vanish outside K, and if in addition, $\phi_n \to \phi$ in the topology of $C_0^\infty(\Omega)$ we shall write

$$\phi_n \overset{\to}{\to} \phi \text{ in } \Omega.$$

ii. A linear functional F defined on $C_0^\infty(\Omega)$ such that $F(\phi_n) \to F(\phi)$ whenever $\phi_n \overset{\to}{\to} \phi$ in Ω is called a distribution in Ω.

iii. The family of all distributions in Ω is denoted by $D(\Omega)$.

Definition VI-2. Let Ω be an open set in R^n. Let f be a function defined in Ω which is (Lebesgue) integrable over every compact subset of Ω, then the distribution F, defined by

$$F(\phi) = \int_{\Omega} \phi\ (\underline{x})\ f\ (\underline{x})\ d\ \underline{x} \qquad \phi\ \varepsilon\ C_0^{\infty}(\Omega)$$

is called the distribution corresponding to f.

If a distribution corresponds to two functions f and g, then f = g almost everywhere.

Definition VI-3. A distribution F which corresponds to a function f in the sense of Definition VI-2 is said to be a function. If f is continuous, differentiable, belongs to $L^2(\Omega)$, $C^k(\Omega)$, $C_0^{\infty}(\Omega)$, etc., F will be said to be continuous, differentiable, belong to $L^2(\Omega)$, $C^k(\Omega)$, $C_0^{\infty}(\Omega)$, etc. respectively.

Thus a distribution, which is a functional, is identified with the function to which it corresponds. The introduction of distributions enables one, as will be seen later, to formulate a partial differential operator in a complete function space, in particular a Hilbert space. The following definition shows how a distribution may be differentiated partially.

Definition VI-4. Let τ be a formal partial differential operator defined in an open subset Ω of R^n, and with real coefficients in $C^{\infty}(\Omega)$. Let F be a distribution in Ω. Then τ F will denote the distribution defined by the equation

$$(\tau F)(\phi) = F(\tau^*\phi) \qquad \phi\ \varepsilon C_0^{\infty}(\Omega).$$

The fact that $\phi_n \underset{n\to}{\to} \phi$ implies $\tau^*\phi_n \underset{n\to}{\to} \tau^*\phi$ is the justification for this definition. Additional justification for Definition VI-4 is provided by the statements of the following lemma:

Lemma VI-1. Let Ω be a subset of R^n.

i. If the distribution F in Ω corresponds to the function f in $C^n(\Omega)$ and if τ is a formal partial differential operator of order at most n defined in Ω, then τF corresponds to τf.

ii. $\tau(\alpha F + \beta G) = \alpha\tau F + \beta\tau G \qquad F, G \, \varepsilon D(\Omega)$

iii. $(\alpha \tau_1 + \beta\tau_2) F = \alpha(\tau_1 F) + \beta(\tau_2 F) \qquad F \, \varepsilon D(\Omega)$

iv. $(\tau_1\tau_2)F = \tau_1(\tau_2 F) \qquad F\varepsilon D(\Omega)$.

This lemma provides the justification for addition, multiplication by a scalar, differentiation, etc., of distributions. Additional properties can be found in [29].

6. The Sobolev Space $H^k(\Omega)$

The Sobolev space $H^k(\Omega)$ constitutes an important class of subspaces of $D(\Omega)$, which are at the same time spaces of functions. The following lemma is elementary:

Lemma VI-2. Let Ω be an open subset of R^n and let F be in $D(\Omega)$. Then F is a function in $L^2(\Omega)$ if and only if there exists a finite constant K such that

$$|F(\phi)| \leqq K||\phi||_2, \quad \phi \, \varepsilon C_0^\infty(\Omega)$$

where $||\phi||_2^2 = \int_\Omega \phi^2 \, d\, \underline{x}$ with the integral taken in the Lebesgue sense.

Definition VI-5. Let Ω be an open subset of R^n and let k be a non-negative integer. Then

i. the set of all F in $D(\Omega)$ such that $\partial^J F$ is in $L^2(\Omega)$ for all $|J| \leqq k$ will be denoted by $H^k(\Omega)$. For each real pair F,G in $H^k(\Omega)$ we write

$$< F,G>_k = \sum_{|J| \leqq k} \int_\Omega (\partial^J F)(\underline{x})(\partial^J G)(\underline{x})d\underline{x}$$

and

$$||F||_k = (< F,F>_k)^{1/2}. \qquad \text{(VI-2)}$$

ii. the symbol $H_0^k(\Omega)$ will denote the closure in the norm of $H^k(\Omega)$ of the $C_0^\infty(\Omega)$ functions.

Lemma VI-3. Let Ω be an open set in R^n. Then the space $H^k(\Omega)$ of the preceding definitions is a complete Hilbert space, and the space $H_0^k(\Omega)$ is a closed subspace of $H^k(\Omega)$. Moreover:

$$H_0^0(\Omega) = H^0(\Omega) = L^2(\Omega)$$

$$H^{k+1}(\Omega) \subseteqq H^k(\Omega) \qquad k \geq 0$$

$$H_0^{k+1}(\Omega) \subseteqq H_0^k(\Omega) \qquad k \geq 0.$$

Lemma VI-4. Let τ be a formal partial differential operator of order k and let $j > k$. Let Ω_0 be a proper subset of Ω, an open set in R^n, such that $\bar{\Omega}_0$ is compact and let $a_J(\underline{x})$ belong to $C^\infty(\Omega)$, then $F \to \tau F$ where $F \varepsilon H^j(\Omega_0)$ implies $\tau F \varepsilon\ H^{j-k}(\Omega_0)$ and the mapping is continuous.

$F\ \varepsilon\ H_0^j(\bar{\Omega}_0)$ implies $\tau F\ \varepsilon\ H_0^{j-k}(\bar{\Omega}_0)$ and the mapping is continuous.

7. Sobolev Imbedding Theorem

Definition VI-6. Let p be a point of the subset A of R^n. Then A is said to be smooth in the vicinity of p if there exists a neighborhood U of P and a mapping ϕ of U on a spherical neighborhood V of the origin such that

i. ϕ is one-to-one, ϕ is infinitely often differentiable, and ϕ^{-1} is infinitely often differentiable.

ii. $\phi(AV) = V \cap \{\underline{x}\ \varepsilon\ R^n | \underline{x}_1 = 0\}$.

If the set A is smooth in the vicinity of each of its points, it is said to be smooth, or to be a smooth surface.

Theorem VI-1. Let Ω be a bounded set in R^n. Suppose that $\partial\Omega$ is a smooth surface and that no point of the boundary of Ω is interior to $\bar{\Omega}$. If $2k > n$ and $0 < m < k - \frac{n}{2}$ then every derivative of order not more than m of the distribution $F\ \varepsilon\ H^k(\Omega)$ is continuous and the imbedding operator from $H^k(\Omega)$ into $C^m(\Omega)$ is bounded and completely continuous.

8. Differential Operator Representation

In order to apply the stability theorems of Chapter V, the formal partial differential operator must be extended to be defined in a Hilbert space, i.e., in a complete normed linear space, with the norm induced by an inner product. The important requirement is that the space of differentiable functions is complete, i.e., that the operator, its domain and its range are defined in such a way that all required formal mathematical operations are valid within the setting of the space.

Let τ be a formal partial differential operator defined in a domain Ω of R^n, then $T_0(\tau)$ and $T_1(\tau)$ will denote the operators in $L^2(\Omega)$ defined by the equations

$$\mathcal{D}(T_0(\tau)) = C_0^\infty(\Omega) \ ; \ T_0(\tau)f = \tau f \qquad f \ \varepsilon \mathcal{D}(T_0(\tau)).$$

$$\mathcal{D}(T_1(\tau)) = \{f \ \varepsilon \ D(\Omega) | \ f\varepsilon L^2(\Omega), \ \tau f \varepsilon L^2(\Omega)\}; \qquad \text{(VI-3)}$$

$$T_1(\tau) \ f = \tau \ f \qquad f \ \varepsilon \ \mathcal{D}(T_1(\tau)).$$

Then by Definition VI-1 there follows that $T_1(\tau) = (T_0(\tau^*))^*$. Thus by Corollary III-1, $T_1(\tau)$ is a closed operator in $L^2(\Omega)$. Moreover by Lemma VI-1 $T_0(\tau) \subseteq T_1(\tau)$ is a closed extension of $T_0(\tau)$.

Thus $T_1(\tau)$ operating on an element of $\mathcal{D}(T_1(\tau)) \subseteq L^2(\Omega)$ will result in an element in $L^2(\Omega)$. $L^2(\Omega)$ is a complete linear space. The proper Hilbert space setting is obtained by defining a norm, $||\cdot||_{L^2} = ||\cdot||_0$, on $L^2(\Omega)$ based on the inner product:

$$< f,g>_0 = \int_\Omega f \ g \ d \ \underline{x} \qquad f, \ g \ \varepsilon \ L^2(\Omega).$$

and

$$||f||_0 = (< f,f>_0)^{1/2}. \qquad \text{(VI-4)}$$

In order to derive the unbounded operator A from $T_1(\tau)$ a domain and range in the Hilbert space $L^2(\Omega)$ must be specified. The requirement of the existence of bounded derivatives and the necessity for a proper adjoint relationship for A in the Hilbert space setting dictates a linear subset of $L^2(\Omega)$. These requirements can be met by specifying the domain as a Sobolev space.

For a formal partial differential operator, this could lead to a specification for A as: the operator in the Hilbert space $L^2(\Omega)$ defined by:

$$\mathcal{D}(A(\tau)) = \mathcal{D}(A) = \mathcal{D}(T_1(\tau)) \cap H_0^k(\Omega) \cap H^{2k}(\Omega)$$

$$A\, f = T_1(\tau)\, f \quad , \quad f \in \mathcal{D}(A).$$

Under these conditions $\mathcal{R}(A)$ will also be in $L^2(\Omega)$.

Another way of achieving the adjoint relationship for A is by defining generalized derivatives in the sense that integration by parts formally carries through[30]. In both cases it places restrictions on the values of the distributional derivatives or derivatives on the boundary. Thus if $\bar{\Omega}=\Omega + \partial\,\Omega$ has a sufficiently smooth boundary, $\partial\,\Omega$, then the space $H_0^k(\Omega)$ has (distributional) derivatives up to order k-1 which approach zero as the boundary is approached in Ω.

A more detailed discussion concerning this formulation of differential operators in the appropriate spaces can be found in Dunford and Schwartz[29]. This development allows us to apply the stability theory developed for operator differential equations in Chapter V to a class of partial differential equations that can be placed in the above Hilbert space setting.

B. Selection of Norms

It has been pointed out that stability is defined with respect to a norm. However, it is obvious that in formulating the particular stability problem the norm cannot be chosen at random. In many instances one is able to select a norm in a natural way by considering the physical properties of the system. This leads for the wave equation to a consideration of the energy which provides as the natural norm the L^2-norm as defined by (VI-4) i.e., a Hilbert space setting.

One the other hand, for the heat equation the natural norm is given by the supremum of the temperature, thus (VI-1) with $J = 0$. Other examples can be given where the natural norm is not induced by an inner product or does not satisfy the parallelogram law.

The stability theory is based on the Hilbert space theory of dissipative operators, the generators of contraction semi-groups or groups. Thus the Hilbert space structure is essential for the stability theory.

The specific problem considered here concerns stability with respect to Hilbert space norm. Whenever the natural norm does not correspond to an inner product one can only draw conclusions about the stability properties with respect to this norm if it is equivalent with the particular Hilbert space norm for which the stability is determined.

The role Sobolev's Imbedding Theorems might play in relating stability properties under different norms, not necessarily equivalent, is suggested for future research.

VII. STABILITY OF AN EVOLUTION EQUATION

In the previous chapter it was pointed out that in order to define $\underset{\sim}{A}$ as an operator on a complete space, it is necessary to introduce distributions. $\underset{\sim}{A}$ is then defined on some Sobolev space. $\underset{\sim}{A}$ is thus associated with generalized functions and the conditions for stability of the solutions of (V-1) must be interpreted as stability conditions for the distribution solutions.

The Sobolev Imbedding Theorem makes it possible to determine if and when the stability properties can be extended to the C^j-norms.

The Dirichlet boundary value problem for strongly elliptic partial differential operators will be studied to demonstrate this development. Section A is devoted to some basic properties of strongly elliptic formal partial differential operators. The basic theorems concerning the formulation of the Dirichlet problem in the distribution sense conclude this section.

The actual stability problem is solved in Section B.

A. Elliptic Partial Differential Operator

Definition VII-1. Let

$$\tau = \sum_{|J| \leq p} a_J(\underline{x})\, \partial^J$$

be a formal partial differential operator of order p with real coefficients defined in a domain Ω of R^n. Then if for each nonzero vector $\underline{\xi} \in R^n$ we have

$$\sum_{|J|=p} a_J(\underline{x})\, \underline{\xi}^J \neq 0 \qquad \underline{x} \in \Omega$$

the operator τ is said to be elliptic.

The boundary value problem to be discussed considers the strongly elliptic partial differential operators, i.e., those operators of even order for which Garding's Inequality holds.

Lemma VII-1. Let τ be an elliptic operator of even order 2p with real coefficients defined in a domain Ω_0 of R^n. Let Ω be a bounded open set such that $\bar{\Omega} \subseteq \Omega_0$. Let

$$\tau = \sum_{|J| \leq 2p} a_J(\underline{x})\, \partial^J \tag{VII-1}$$

and suppose that

$$(-1)^p \sum_{|J|=2p} a_J(\underline{x})\, \underline{\xi}^J > 0, \ \underline{x} \in \Omega_0, \ \underline{\xi} \neq \underline{0}, \ \underline{\xi} \in R^n. \tag{VII-2}$$

Then there exist constants $K < \infty$ and $k > 0$ such that

$$< \tau f, f>_0 + K< f,f>_0 \geq k\, ||f||_p^2 \qquad f \in C_0^\infty(\Omega), \tag{VII-3}$$

or $<(\tau + KI)\, f,f>_0 \geq k\, ||f||_p^2$. For K=0 we get

$$< \tau f,f>_0 \geq k||f||_p^2 = k< f,f>_p$$

or

$$< -\tau f,f>_0 \leq -\, k\, < f,f>_p \leq -\, k< f,f>_0.$$

τ is called strongly elliptic and (VII-3) is known as Garding's Inequality.

The next step is to specify a differential operator for τ(still without boundary conditions) which later might be a candidate for the closed extension of an operator associated with the Dirichlet problem. $T_1(\tau)$ is the closed operator defined by (VI-3).

Theorem VII-1. Let τ be an elliptic formal partial differential operator of even order 2p, satisfying the hypothesis of Lemma VII-1, and defined in a bounded domain Ω in R^n which satisfies the hypothesis of the lemma. Let $T = T(\tau)$ be the operator in the Hilbert space $L^2(\Omega)$ defined by the equations

$$\mathcal{D}(T(\tau)) = \mathcal{D}(T) = \mathcal{D}(T_1(\tau)) \cap H_0^p\ (\Omega)) \tag{VII-4}$$

$$Tf = T_1(\tau)f, \ f \in \mathcal{D}(T).$$

Then $\sigma(T)$, the spectrum of T, is a countable discrete set of points with no finite limit points and for $\lambda \notin \sigma(T)$, $R(\lambda; T)$ is a compact operator.

The restriction of the domain of T to a subspace of the domain of $T_1(\tau)$ by (VII-4) makes $R(\lambda; T)$ a compact operator.

Corollary VII-1. Let the hypotheses of Theorem VII-1 be satisfied. Then there exists a constant $K < \infty$ and a constant $k > 0$ such that

$$< Tf, f>_0 + K < f, f>_0 \geqq k \, ||f||_p^2 , \quad f \in \mathcal{D}(T) \qquad \text{(VII-5)}$$

The next step is to specify the adjoint.

Theorem VII-2. Let τ be an elliptic formal partial differential operator of even order 2p, defined in a bounded domain Ω. Suppose the hypotheses of Lemma VII-1 are satisfied. Let T and S be operators in the Hilbert space $L_2(\Omega)$ defined by

$$\mathcal{D}(T) = \mathcal{D}(T_1(\tau)) \cap H_0^p(\Omega), \; \mathcal{D}(S) = \mathcal{D}(T_1(\tau^*)) \cap H_0^p(\Omega)$$

$$Tf = T_1(\tau)f, \; f \in \mathcal{D}(T); \; Sf = T_1(\tau^*)f, \; f \in \mathcal{D}(S).$$

Then $T = S^*$ and $S = T^*$.

Moreover we have thus $< Tf, g>_0 = < f, Sg>_0$ for any $f \in \mathcal{D}(T)$ and $g \in \mathcal{D}(S)$.

The Dirichlet problem can be formulated as follows:

Definition VII-2. Let Ω be a domain in R^n whose boundary $\partial\Omega$ contains a part Γ which is a smooth surface. Suppose that no point in Γ is interior to $\bar{\Omega}$. Let k be a positive integer. Then if f is in $C^{k-1}(\bar{\Omega})$ and $\partial^J f(\underline{x})$ vanishes for all $\underline{x} \in \Gamma$ and all J with $|J| \leqq k - 1$ we will say that f satisfies the Dirichlet condition of order k on Γ or that f and its first k-1 normal derivatives vanish on Γ and we write:

$$(\partial_\nu(\Gamma))^j f(\underline{x}) = 0 \quad \underline{x} \in \Gamma, \; 0 \leqq j \leqq k-1 \tag{VII-6}$$

Remark: The subscript ν in the preceding formula indicates the derivative in the direction of the normal to Γ. If Γ is a closed rectangular (n-1)-dimensional hypersurface in R^n with side Γ_i perpendicular to the x_i-coordinate axis then (VII-6) becomes

$$\frac{\partial^j f(\underline{x})}{\partial x_i^j} = 0 \qquad \underline{x} \in \Gamma_i \qquad \begin{matrix} 0 \leqq j \leqq k-1 \\ 1 \leqq i \leqq n. \end{matrix}$$

The following theorems will identify the operator τ with Dirichlet boundary conditions with an appropriate operator $\underline{A}$ defined on distributions and thus closed. The notation $T(\tau)$ is used for the operator in the Hilbert space $L^2(\Omega)$ defined by the equations

$$\begin{aligned} \mathcal{D}(T(\tau)) &= H_0^p(\Omega) \cap H^{2p}(\Omega) \\ T(\tau)f &= \tau f \;;\; f \in \mathcal{D}(T(\tau)). \end{aligned} \tag{VII-7}$$

From Theorem VII-2 and the fact that $H_0^p(\Omega) \cap H^{2p}(\Omega)$ is a dense subset of $L^2(\Omega)$, it follows that $T(\tau)$ is a closed operator.

Theorem VII-3. Let τ be a strongly elliptic formal partial differential operator as given by (VII-1) and satisfying (VII-2). Let Ω be a bounded subdomain $\bar{\Omega} \subseteqq \Omega_0$. Let $\partial\Omega$ be a smooth surface and let no point in $\partial\Omega$ be interior to the closure of Ω. Let T and $\hat{T}$ be operators in the Hilbert space $L^2(\Omega)$ defined by the equations:

$$\begin{aligned} \mathcal{D}(T) = \mathcal{D}(\hat{T}) &= \{f \in C^\infty(\bar{\Omega}) \mid f(\underline{x}) = \\ &= \partial_\nu(\partial\Omega)\, f(\underline{x}) = \ldots = \partial_\nu^{p-1}(\partial\Omega)\, f(\underline{x}) = 0, \underline{x} \in \partial\Omega\} \end{aligned} \tag{VII-8}$$

$$Tf = \tau f \;,\; \hat{T}f = \tau^* f, \quad f \in \mathcal{D}(T) = \mathcal{D}(\hat{T}).$$

Then let W and $\hat{W}$ denote the operators whose graphs are the closures of the graphs of T and $\hat{T}$ respectively. Then

i. $W^* = \hat{W}$, $\hat{W}^* = W$.

ii. $\sigma(W)$ is a countable discrete set of points with no finite limit point.

iii. If $\lambda \notin \sigma(W)$, $R(\lambda; W)$ is a compact operator.

iv. If $\lambda \notin \sigma(W)$, $R(\lambda; W)$ is a continuous mapping on $H^m(\Omega)$ into $H^{m+2p}(\Omega)$ for every $m \geqq 0$.

v. If Wf is in $H^m(\Omega)$ and if $m \geqq \left[\frac{n}{2}\right] - p$, then f is in $C^{p-1}(\bar{\Omega})$ and f satisfies the boundary conditions defining $\mathcal{D}(T)$ stated by (VII-8). ($[\cdot]$ means largest integer, $[5\frac{3}{4}] = 5$).

The operators W and $\hat{W}$ are identical to the operators $T(\tau)$ and $T(\tau^*)$ as defined by (VII-7) respectively. W and $\hat{W}$ are closed operators. The properties of the resolvent of W are indicated in (iii) and (iv).

In the case where $\tau = \tau^*$ a more explicit result follows from the theorem:

<u>Theorem VII-4.</u> Let τ be a strongly elliptic formal partial differential operator as given by (VII-1) and satisfying (VII-2). Let Ω be a bounded subdomain $\bar{\Omega} \subseteqq \Omega_0$. Let $\partial\Omega$ be a smooth surface and let no point in $\partial\Omega$ be interior to the closure of Ω and let $\tau = \tau^*$. Let T be the operator in Hilbert space $L^2(\Omega)$ defined by

$$\mathcal{D}(T) = \{f \in C^\infty(\bar{\Omega}) \mid f(\underline{x}) = \partial_\nu(\partial\Omega)\, f(\underline{x}) = \dots = \partial_\nu^{p-1}(\partial\Omega)\, f(\underline{x}) = 0, \quad \underline{x} \in \partial\Omega\} \qquad \text{(VII-9)}$$

$$Tf = \tau f \ , \ f \in \mathcal{D}(T).$$

Let W be the closure of T, then

i. The operator W is self-adjoint.

ii. The spectrum $\sigma(W)$ is a sequence of points $\{\lambda_n\}$ tending to ∞, and for λ in $\rho(W)$, $R(\lambda; W)$ is a compact operator.

iii. The operator W has a complete countable set $\{\phi_n\}$ of eigenfunctions. Each eigenfunction satisfies the partial differential equations $\tau\ \phi_n = \lambda_n \phi_n$ in Ω, has infinitely many continuous derivatives in the closure of the domain Ω, and satisfies the boundary conditions defining $\mathcal{D}(T)$ of (VII-9).

The results of this theorem are important for the wave equation problem of Chapter VIII.

B. Stability of the Solutions to an Evolution Equation with Strongly Elliptic Partial Differential Operator

The Dirichlet boundary conditions appear in a large class of boundary value problems for elliptic partial differential operators. In stability investigations of systems represented by partial differential equations, the stability analysis is generally carried out for systems perturbed from equilibrium. In this case one can often introduce zero boundary conditions.

The formulation of the general conditions for the stability and existence of solutions to the class of strongly elliptic partial differential operators will cover a large class of systems.

Consider the system as formulated by (II-1), thus

$$\frac{\partial u}{\partial t} = L\ u = -\ \tau\ u \qquad \text{(VII-10)}$$

where τ is now a time-invariant strongly elliptic formal partial differential operator of even order 2p with real coefficients defined in a domain Ω_0 of R^n,

$$\tau = \sum_{|J| \leq 2p} a_J(\underline{x})\, \partial^J . \tag{VII-11}$$

Then in any bounded open domain Ω, such that $\bar{\Omega} \subseteq \Omega_0$, Garding's Inequality holds for all $u \in C_0^\infty(\Omega)$:

$$< \tau u, u>_0 + K < u,u>_0 \geq k < u,u>_p \tag{VII-12}$$

for some $K < \infty$ and $k > 0$.

Next we want to associate with L and thus τ a Dirichlet boundary value problem. Hence let Ω be a bounded subdomain of Ω_0 with $\bar{\Omega}$ interior to Ω_0. Let $\partial\Omega$ be a smooth surface with no point of $\partial\Omega$ interior to $\bar{\Omega}$. Then (VII-10) defines an operator differential equation

$$\frac{d\,u}{dt} = -\,T\,u\ , \quad u \in \mathcal{D}(T) \tag{VII-13}$$

$$\mathcal{D}(T) = \{u \in C^\infty(\bar{\Omega}) \mid u = \partial_\nu(\partial\Omega)u = \ldots = \partial_\nu^{p-1}(\partial\Omega)\, u = 0;\ \underline{x} \in \partial\Omega\} \tag{VII-14}$$

$$T\,u = \tau u\ ;\ u \in \mathcal{D}(T).$$

The stability theory developed in Chapter V cannot be applied to (VII-13) even though it is an operator differential equation. In order to get (V-1) T must be defined on a complete space with suitable norm. Introduction of distributions and Theorems VII-3 provides the appropriate differential operator W, the closure of T with respect to $L^2(\Omega)$, as

$$W\,f = \tau f \quad ; \quad f \in \mathcal{D}(W)$$

$$\mathcal{D}(W) = H_0^p(\Omega) \cap H^{2p}(\Omega) \subseteq L^2(\Omega). \tag{VII-15}$$

And the corresponding operator differential equation becomes

$$\frac{d\,f}{dt} = -\,W\,f \qquad\qquad f \in \mathcal{D}(W). \tag{VII-16}$$

Furthermore $R(W) \subseteq L^2(\Omega)$ and since $H_0^p(\Omega)$ is dense in $L^2(\Omega)$ $\overline{\mathcal{D}(W)} = L^2(\Omega)$, thus (VII-16) is an operator differential equation, associated with (VII-10), which belongs to the class (V-1).

Theorem VII-3 enables us to define the adjoint of W, W^*, as the closure $\hat{W}$ of $\hat{T}$ where $\mathcal{D}(T) = \mathcal{D}(\hat{T})$ and $\hat{T}\, u = \tau^*\, u$ for $u \in \mathcal{D}(\hat{T})$. Since τ is strongly elliptic τ^* will also be strongly elliptic, thus satisfying Garding's Inequality for $u \in C_0^\infty(\Omega)$. Replacing τ by τ^* in Theorem VII-3 gives $\mathcal{D}(W) = \mathcal{D}(W^*) = H_0^p(\Omega) \cap H^{2p}(\Omega) = \mathcal{D}(\hat{W})$. This is an important property, to be used in the following stability theorem. Notice that if $\mathcal{D}(W) = \mathcal{D}(W^*)$, then W is self-adjoint if $\tau = \tau^*$

It should also be noted that the imbedding of the closed subspace $H_0^p(\Omega)$ in $L^2(\Omega)$ implies that there exists a constant $C \geqq 1$ such that

$$< f,f>_p \geqq C < f,f>_0 \quad \text{for all } f \in H_0^p(\Omega). \qquad \text{(VII-17)}$$

<u>Theorem VII-5</u>. Let τ be a strongly elliptic formal partial differential operator as defined by (VII-11), satisfying the Garding Inequality (VII-12). Let T of (VII-14) be the operator associating τ with a Dirichlet boundary value problem, and let W be the closure of T as defined by (VII-15).
Then the null solution of

$$\frac{d\,f}{dt} = -\,W\,f \qquad f \in \mathcal{D}(W) = H_0^p(\Omega) \cap H^{2p}(\Omega) \qquad \text{(VII-18)}$$

is asymptotically stable with respect to the L^2-norm if a $C_0 \geqq 1$ can be found such that

$$\text{i.} \qquad ||f||_p^2 \geqq C_0 ||f||_0^2 \qquad f \in \mathcal{D}(W) \qquad \text{(VII-19)}$$

and

$$\text{ii.} \qquad C_0 k - K > 0 \qquad \text{(VII-20)}$$

where k and K are two constants satisfying Garding's Inequality for τ.

<u>Proof</u>: Define the symmetric, bilinear, positive definite functional $V(f,g)$ by

$$V(f,g) = \langle f,g\rangle_0 \quad \text{for} \quad f,g \in L^2(\Omega).$$

The Lyapunov Functional $v(f)$ is thus defined by

$$v(f) = V(f,f) = \langle f,f\rangle_0 \quad \forall\, f \in L^2(\Omega).$$

Then $\dot{v}(f) = 2V(-Wf,f) = 2\langle -Wf,f\rangle_0 \qquad f \in \mathcal{D}(W).$ (VII-21)

Since τ satisfies (VII-12) we have by Corollary VII-1:

$$\langle Wf,f\rangle_0 + K\langle f,f\rangle_0 \geqq k \langle f,f\rangle_p \qquad \forall\, f \in \mathcal{D}(W)$$

and with (VII-19):

$$\langle Wf,f\rangle_0 \geqq (C_0k - K) \langle f,f\rangle_0 \qquad \forall\, f \in \mathcal{D}(W) \tag{VII-22}$$

W is closed, and so is $-W$, $\overline{\mathcal{D}(W)} = \overline{\mathcal{D}(-W)} = L^2(\Omega)$.

Since $\mathcal{D}(W) = \mathcal{D}(W^*)$, it follows from (VII-22):

$$\langle W^*f,f\rangle_0 = \langle f,Wf\rangle_0 = \langle Wf,f\rangle_0 \geqq (C_0k - K) \langle f,f\rangle_0 \tag{VII-23}$$

for all $f \in \mathcal{D}(W^*) = \mathcal{D}(W)$. For $(C_0k - K) > 0$, W is a one-to-one mapping from $H_0^p(\Omega) \cap H^{2p}(\Omega)$ into $L^2(\Omega)$ and $R(W)$ is closed. $H_0^p(\Omega) \cap H^{2p}(\Omega)$ is reflexive[23], thus W^* maps $H_0^p(\Omega) \cap H^{2p}(\Omega)$ into $L^2(\Omega)$ and since $N(W^*) = R(W)^{\perp}$, the annihilator of $R(W)$ in $L^2(\Omega)$, it follows from the closedness of $R(W)$ that $R(W) = N(W^*)^{\perp}$. However, by (VII-23) also W^* is one-to-one and $N(W^*) = \{0\}$, thus $R(W) = L^2(\Omega)$. Substitution of (VII-23) into (VII-21) gives

$$\dot{v}(f) = 2 \langle -Wf,f\rangle_0 \leqq -2(C_0k - K) \langle f,f\rangle_0 \quad \forall f \in \mathcal{D}(W). \tag{VII-24}$$

Thus for $C_0k-K > 0$ the operator $-W$ with $\overline{\mathcal{D}(-W)} = L^2(\Omega)$ is strictly dissipative and $R(I+W) = L^2(\Omega)$. Hence by Theorem V-5 the Lyapunov Functional $v(f)=\langle f,f\rangle_0$ assures the asymptotic stability of the null solution of (VII-18). Moreover,

$-W$ generates a semi-group $\{T_t \ ; \ \ t \geqq 0\}$ in $L^2(\Omega)$ with

$$||T_t||_0 \leqq e^{-\alpha t} \qquad t \geqq 0,$$

where $\alpha = C_0 k-K$.

In the stability analysis the objective becomes thus to reduce $<-Wf,f>_0$ to the form $-\alpha< f,f>_0$. The choice of the maximal C_0 which satisfies (VII-19) is very important. Integral inequalities[(31)] are available to facilitate this reduction. These integral inequalities are to some extent based on an estimate of the eigenvalues. In Chapter IX it will be shown that a proper selection of v(f) can improve considerably the effectiveness of the use of these inequalities.

A special case arises when $\tau = \tau^*$. Then it follows from Theorem VII-4 that the spectrum of W, uniquely determined since $W = W^*$, is a sequence of points $\{\lambda_n\}$ tending to ∞. Let λ_{min} be the smallest λ_n; then

$$\lambda_{min}< f,f>_0 \leqq < Wf,f>_0 \quad \forall\ f \in \mathcal{D}(W).$$

Thus, if $\lambda_{min} > 0$ we get

$$< - Wf,f>_0 \leqq - \lambda_{min} < f,f>_0 < 0 \quad \forall\ f \in \mathcal{D}(W)$$

and $-W$ is strictly dissipative. However, the determination of the eigenvalues is generally not easy, especially when the coefficients of τ depend on the system parameters. In many such cases integral inequalities might be used more easily.

The question can be raised if similar results can be derived for the asymptotic stability of the null solution to (VII-16) with respect to the H^m-norm. This question becomes important when the Sobolev Imbedding Theorem is used to deduce stability properties with respect to a C^j-norm.

From Theorem VII-3 follows that the closure of T in H^{2m} is W_{2m}. For

Garding's Inequality to hold in $H^{2m}(\Omega)$ it appears that $\mathcal{D}(W_{2m})$ must be restricted to

$$\mathcal{D}(W_{2m}) = H_0^{p+m}(\Omega) \cap H^{2p+2m}(\Omega) \subseteqq L^2(\Omega)$$

and giving for all $f \varepsilon \mathcal{D}(W_{2m})$

$$< W_{2m}f,f>_{2m} + K< f,f>_{2m} \geqq k < f,f>_{2m+p}. \qquad \text{(VII-25)}$$

This gives rise to the following conjecture:

Conjecture VII-1. Let W_{2m} be the closure of T, as defined by (VII-14), with respect to $H^{2m}(\Omega)$. Then the null solution of

$$\frac{d\ f}{dt} = - W_{2m}\ f \qquad f\varepsilon\ \mathcal{D}(W_{2m}) = H_0^{p+m}(\Omega) \cap H^{2p+2m}(\Omega)$$

is asymptotically stable with respect to the H^{2m}-norm if there exists a $C_{2m} \geqq 1$, satisfying

i. $\qquad ||f||_{p+2m}^2 \geqq C_{2m}||f||_{2m}^2 \qquad \forall\ f\ \varepsilon\ \mathcal{D}(W_{2m})$

and

ii. $\qquad C_{2m}\ k - K > 0$

where k and K are the constants satisfying Garding's Inequality for τ.

The derivation of this result is analogous to the proof of Theorem VII-5.

Then for sufficiently large $2m \geqq [\frac{n}{2}] + j + 1$ it follows from Sobolev's Imbedding Theorem that $f\ \varepsilon\ H^{2m}(\Omega)$ implies $f\ \varepsilon C^j(\bar{\Omega})$ and

$$||f||_{C^j(\Omega)} \leqq c||f||_{2m}$$

where c is a positive constant.

However, for some initial function $f_0 \varepsilon \mathcal{D}(W_{2m}) = H_0^{p+m}(\Omega) \cap H_0^{2p+2m}(\Omega)$, $\underline{T}_t f_0$ remains in $\mathcal{D}(W_{2m})$ for $t \geqq 0$[23], $\underline{T}_t$ is the semi-group generated by W_{2m}. Thus for sufficiently large $2m \geqq [\frac{n}{2}] - 2p + j + 1$ it follows by Sobolev's Imbedding Theorem, that $\underline{T}_t f_0 \ \varepsilon \ H^{2m+p}(\Omega)$ implies $\underline{T}_t f_0 \ \varepsilon \ C^j(\bar{\Omega})$ and

$$||\underline{T}_t f_0||_{C^j(\bar{\Omega})} \leqq c_1 ||\underline{T}_t f_0||_{2m+2p}$$

where c_1 is some positive constant. Thus any asymptotically stable semi-group trajectory in $H^{2m+2p}(\Omega)$ should be asymptotically stable in $C^j(\bar{\Omega})$.

Next consider as a specific example a diffusion equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} - b\, u \; ; \quad t \geqq 0 \tag{VII-26}$$

where c is a positive constant and b is a constant, $0 \leqq x \leqq 1$. Assume the boundary conditions to be $u(t,0) = u(t,1) = 0$. (VII-26) is thus formulated as the Dirichlet boundary value problem for a second order partial differential equation defined on $[0,1]$ of R^1. The results are compared in Chapter VIII with those obtained for the case where $L = -c \frac{\partial^2}{\partial x^2} + b$ is taken in the wave equation representation.

Here $\tau = -c \frac{\partial^2}{\partial x^2} + b = \tau^*$ is a strongly elliptic formally self-adjoint partial differential operator defined on $[0,1]$.

(VII-26) is associated with the closed self-adjoint operator differential equation of class (V-1):

$$\frac{d\,f}{dt} = -\,W\,f \qquad\qquad f \ \varepsilon \ \mathcal{D}(W)$$

$$\mathcal{D}(W) = H_0^1([0,1]) \cap H^2([0,1]) \tag{VII-27}$$

$$W\,f = \tau f \qquad\qquad f \ \varepsilon \ \mathcal{D}(W).$$

Take as Lyapunov Functional:

$$v(f) = \langle f,f \rangle_0 = ||f||_0^2 \qquad f \in \mathcal{D}(W)$$

evaluating $\dot{v}(f)$ on the $C_0^\infty([0,1])$ functions

$$\dot{v}(f) = 2\langle Lf,f\rangle_0 = 2\int_0^1 \{c\frac{\partial^2 f}{\partial x^2} \cdot f - b f^2\} d x = -2\int_0^1 \{ c (\frac{\partial f}{\partial x})^2 +$$

$$+ b f^2\} d x.$$

This last expression can be reduced with the integral inequality[31]

$$\int_0^1 (\frac{\partial f}{\partial x})^2 d x \geqq \pi^2 \int_0^1 f^2 d x$$

to

$$\dot{v} \leqq -2(c \pi^2 + b) \int_0^1 f^2 d x = -2(c \pi^2 + b) \langle f,f\rangle_0.$$

A sufficient condition for the asymptotic stability of the null solution of (VIII-27) is thus

$$c \pi^2 + b > 0.$$

Or since for the strong ellipticity of τ, $c > 0$ is required:

$$c > 0 \quad \text{and} \quad b > - c \pi^2.$$

Thus $-W$ generates a semi-group $\{ \underline{T}_t ; t \geqq 0\}$ in $L^2([0,1])$ which is bounded by:

$$||\underline{T}_t||_0 \leqq e^{-\alpha t} \qquad t \geqq 0,$$

where $\qquad \alpha = b + \pi^2 c.$

The results obtained are identical to those obtained by evaluating the eigenvalues of $-\tau$. These are

$$\lambda_n = b + cn^2 \pi^2 \qquad (n = \pm 1, \pm 2, \ldots.)$$

Negative b are allowed as long as $b > -c \pi^2$, since $c (\frac{\partial f}{\partial x})^2 + bf^2 = 0$, implies for $f(0) = f(1) = 0$ that $f = 0$ is the only solution.

VIII. STABILITY OF WAVE EQUATIONS

In this chapter the stability properties of an important class of partial differential equations will be investigated. Wave equations appear frequently in the mathematical representations of physical systems. The development of a Lyapunov Functional for this system is sketched in Section A. Specific applications are given in Section B. The formulation of the Lyapunov Functional and its derivative show that the system possesses the group property rather than the semi-group property, and thus necessary and sufficient conditions for stability can be established.

A. Development of a Lyapunov Functional

The following formal derivation of a suitable Lyapunov Functional, not mathematically rigorous, constitutes an important step in solving the problem of asymptotic stability for the wave equation with Lyapunov's Direct Method. Let us consider the simple wave equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + b\,u - c\,\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(VIII-1)}$$

with a, b, and c positive constants. Let x be normalized to the interval [0,1] and assume $u(t,0) = u(t,1) = 0$. Under these assumptions one expects the null solution $u = 0$ to be asymptotically stable. However, if the total energy of the system is chosen as Lyapunov functional

$$v(u) = \int_0^1 \{(\frac{\partial u}{\partial t})^2 + b\,u^2 + c(\frac{\partial u}{\partial x})^2\}d\,x \qquad \text{(VIII-2)}$$

then, with the assumption that the derivative can formally be determined:

$$\dot{v}(u) = -2a\int_0^1 (\frac{\partial u}{\partial t})^2\,d\,x.$$

Since $\dot{v}(u)$ is only negative semi-definite, the null solution $u = 0$ is assured to be stable but not asymptotically stable.

A similar result follows if one interprets (VIII-1) in the context of a dissipative system [24]. Here (VIII-1) is written as

$$\frac{\partial \underline{u}}{\partial t} = \underline{L}\,\underline{u}$$

with

$$\underline{u} = \begin{bmatrix} \sqrt{b}\, u \\ \sqrt{c}\, u_x \\ u_t \end{bmatrix} \quad ; \quad \underline{L} = \begin{bmatrix} 0 & 0 & \sqrt{b} \\ 0 & 0 & \sqrt{c}\,\frac{\partial}{\partial x} \\ -\sqrt{b} & \sqrt{c}\,\frac{\partial}{\partial x} & -a \end{bmatrix}$$

Thus the choice of (VIII-2) as Lyapunov Functional is apparently not the correct one for showing asymptotic stability. The problem is to select a Lyapunov Functional v(u) with a negative definite time derivative.

The procedure for the construction of such a Lyapunov Functional follows to some extent the construction of Lyapunov functions for systems of linear ordinary differential equations [2].

Notice that (VIII-1) can formally be solved by a separation of variables argument [32]. Putting u(t,x) = T(t) X(x) (VIII-1) can be replaced by the system of equations:

$$\frac{d^2T}{dt^2} + a\,\frac{dT}{dt} + \lambda\, T = 0$$

and

$$c\,\frac{d^2X}{dx^2} + b\, X - \lambda\, X = 0 \qquad \text{(VIII-3)}$$

and boundary conditions X(0) = X(1) = 0.

Suppose that the second equation of (VIII-3) is solved and a $\lambda = \lambda_1$ is determined. For this particular example λ_1 will be real, because the differential operator is self-adjoint. Substitution of λ_1 in the first equation of (VIII-3) gives:

$$\frac{d^2T}{dt^2} + a\frac{d\,T}{dt} + \lambda_1 T = 0.$$

This is a second order linear ordinary differential equation with real coefficients that can be written as:

$$\frac{d\,T_1}{dt} = T_2 \qquad\qquad \text{or} \qquad\qquad \frac{d\,\underline{T}}{dt} = \underline{F}\,\underline{T}$$

$$\frac{d\,T_2}{dt} = -\lambda_1 T_1 - a\,T_2$$

The construction of a Lyapunov function for this system follows the usual procedure. Let

$$V(\underline{T}) = \underline{T}^T\,\underline{P}\,\underline{T}$$

where $\underline{P}$ is symmetric and $\underline{T}^T$ denotes the transpose of $\underline{T}$ then

$$\frac{d\,V(\underline{T})}{dt} = \underline{T}^T\,(\underline{F}^T\,\underline{P} + \underline{P}\,\underline{F})\underline{T} = -\,\underline{T}^T\,\underline{Q}\,\underline{T}.$$

The system is asymptotically stable if for a positive definite symmetric matrix $\underline{Q}$, $\underline{P}$ as a solution to the matrix equation

$$\underline{F}^T\,\underline{P} + \underline{P}\,\underline{F} = -\,\underline{Q}$$

is uniquely determined by $\underline{Q}$ and is positive definite symmetric when $\underline{Q}$ is. $\underline{Q}$ is usually taken as the identity matrix. Here the objective is, however, to identify λ_1 with $L = -\,c\frac{\partial^2}{\partial x^2} + b$, and $v(u)$ and $\dot{v}(u)$ should become quadratic forms which are equivalent to the same norm, or can be reduced to equivalent norms. Thus let

$$\underline{Q} = \begin{pmatrix} 2\,a\lambda_1 & 0 \\ 0 & 2\,a \end{pmatrix}$$

then $\underline{P}$ follows as

$$\underline{P} = \begin{bmatrix} 2\lambda_1 + a^2, & a \\ a\,, & 2 \end{bmatrix}$$

The next step is to interpret $\underline{P}$ and $\underline{Q}$ in terms of L. The first consequense is to write (VIII-1) as

$$\frac{d\,\underline{u}}{dt} = L\,\underline{u}$$

where

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u \\ u_t \end{bmatrix}; \quad L = \begin{bmatrix} 0 & , & 1 \\ -L & , & -a \end{bmatrix} .$$

Next v(u) must be patterned after $V(\underline{T}) = \underline{T}^T\,\underline{P}\,\underline{T}$ by replacing λ_1 with L and at the same time taking the L^2-inner product. Thus

$$v(\underline{u}) = \int_0^1 \underline{u}^T \begin{bmatrix} 2L + a^2 & , & a \\ a & , & 2 \end{bmatrix} \underline{u}\,dx \qquad \text{(VIII-4)}$$

Integration by parts and substitution of the boundary conditions gives

$$v(\underline{u}) = \int_0^1 \{2\,c\,(\frac{\partial\,u_1}{\partial\,x})^2 + 2\,b\,u_1^2 + a^2\,u_1 + 2\,a\,u_1 u_2 + 2\,u_2^2\}d\,x. \text{(VIII-5)}$$

With a, b and c > 0, (VIII-5) is equivalent to the norm of the product space $H_0^1([0,1]) \times H_0^0\,([0,1])$.

For the case of self-adjoint L, $\dot{v}(\underline{u})$ can be related to $\dfrac{d\,V(\underline{T})}{dt}$.

In this particular example

$$\dot{v}(u) = -2a \int_0^1 \underline{u}^T \begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix} \underline{u}\,d\,x = -\,2a \int_0^1 \{c\,(\frac{\partial u_1}{\partial\,x})^2 + bu_1^2 + u_2^2\}d\,x. \qquad \text{(VIII-6)}$$

Thus $\dot{v}(\underline{u})$ is equivalent to the norm of the product space $H_0^1([0,1]) \times H_0^0([0,1])$.

For nonself-adjoint operators L (VIII-5) can still be taken as Lyapunov functional, but $\dot{v}(\underline{u})$ must formally be evaluated. An example is the panel flutter problem analyzed by Parks[16]. The Lyapunov functional derived by solving a variational problem is exactly equal to (VIII-4) with L replaced by the appropriate differential operator.

The Lyapunov Functional developed above enables the formulation of the stability problem for the wave equation in terms of the hypotheses of Theorem V-6. Hence we are able to show not only that the solutions are asymptotically stable but exist and satisfy the group property.

Some applications are given in the following section.

B. Application to a Class of Elliptic Partial Differential Operators

Consider the general system equation:

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + L\,u = 0 \tag{VIII-7}$$

with a constant $a > 0$ and $L = \tau$ is a strongly elliptic partial differential operator of even order 2p with real coefficients defined and uniformly bounded in a domain Ω_0 of R^n:

$$\tau = \sum_{|J| \leqq 2p} a_J(\underline{x})\,\partial^J. \tag{VIII-8}$$

and let $\tau = \tau^*$. Then in any open bounded domain Ω such that $\bar{\Omega} \subseteqq \Omega_0$ Garding's Inequality will hold for all $u \in C_0^\infty(\Omega)$:

$$< \tau u,u>_0 + K<u,u>_0 \geqq k<u,u>_p \tag{VIII-9}$$

for some constants $K < \infty$ and $k > 0$.

With L is again associated a boundary value problem. Thus Ω is a bounded subdomain of Ω_0 with $\bar{\Omega}$ interior to Ω_0. Let $\partial\Omega$ be a smooth surface with no point of $\partial\Omega$ interior to $\bar{\Omega}$. Then τ defines a differential operator T such that

$$\mathcal{D}(T) = \{ u \in C^{\infty}(\bar{\Omega}) \; | u = \partial_{\nu}(\partial\Omega)\, u = \ldots \partial_{\nu}^{p-1}(\partial\Omega) u = 0; u \in \partial\Omega)$$

$$T\, u = \tau\, u \quad ; \quad u \in \mathcal{D}(T) \qquad \text{(VIII-10)}$$

T is again not defined on a complete space. In order to get the proper differential operator we introduce distributions and let, by Theorem VII-4, W be the closure of T with respect to $L^2(\Omega)$ so that W is self-adjoint

$$\begin{aligned} \mathcal{D}(W) &= H_0^p(\Omega) \cap H^{2p}(\Omega) \\ W\, f &= \tau f \quad : \quad f \in \mathcal{D}(W). \end{aligned} \qquad \text{(VIII-11)}$$

And the corresponding operator differential equation

$$\frac{d^2 f}{dt^2} + a \frac{d\, f}{dt} + W\, f = 0 \qquad f \in \mathcal{D}(W). \qquad \text{(VIII-12)}$$

From Corollary VII-1 it follows that for all $f \in \mathcal{D}(W)$

$$< W\, f, f>_0 + K < f, f>_0 \geqq k < f, f>_p . \qquad \text{(VIII-13)}$$

Then (VIII-12) can also be written as

$$\frac{d\, \underline{f}}{dt} = \underline{\mathcal{W}}\, \underline{f}, \; \mathcal{D}(\underline{\mathcal{W}}) = H_0^p(\Omega) \cap H^{2p}(\Omega) \times H_0^p(\Omega) \qquad \text{(VIII-14)}$$

where

$$\underline{f} = \begin{bmatrix} f_1 \\ \\ f_2 \end{bmatrix} = \begin{bmatrix} f \\ \\ f_t \end{bmatrix} ; \; \underline{\mathcal{W}} = \begin{bmatrix} 0 & 1 \\ \\ -W & -a \end{bmatrix} \qquad \text{(VIII-15)}$$

Since

$$\underline{\mathcal{W}}\ \underline{f} = \begin{bmatrix} f_2 \\ -W\ f_1 - a\ f_2 \end{bmatrix} \qquad \text{then} \quad \mathcal{R}(\underline{\mathcal{W}}) = H_0^p(\Omega) \times \mathcal{R}(W).$$

Theorem VIII-1. Let W(VIII-11) be the closed self-adjoint differential operator associated with the Dirichlet boundary value problem for the strongly elliptic formlly self-adjoint τ as given by (VIII-8), then the null solution, $\underline{f} = \underline{0}$, of the system

$$\frac{d\ \underline{f}}{dt} = \begin{bmatrix} 0 & 1 \\ -W & -a \end{bmatrix} \underline{f} = \underline{\mathcal{W}}\ \underline{f}\ ;\ \mathcal{D}(\underline{\mathcal{W}}) = H_0^p(\Omega) \cap H^{2p}(\Omega) \times H_0^p(\Omega) \qquad \text{(VIII-16)}$$

is asymptotically stable if $a > 0$ and if there exists a $C_0 > 0$ satisfying

$$\text{i.} \quad ||f_1||_p^2 \geqq C_0 ||f_1||_0^2 \qquad f_1 \in \mathcal{D}(W) \qquad \text{(VIII-17)}$$

and

$$\text{ii.} \quad k - \frac{K}{C_0} > \varepsilon > 0 \qquad \text{(VIII-18)}$$

where $\varepsilon > 0$ and k and K are the constants satisfying Garding's Inequality for τ (VIII-9). $\underline{\mathcal{W}}$ generates a group of exponential type in $H_0^p(\Omega) \times H_0^0(\Omega)$.

Proof: As the derivation in the preceding section shows, define a bilinear functional

$$\hat{V}(\underline{f},\underline{g}) = \sum_{i=1}^{2} \sum_{j=1}^{2} < f_i,\ p_{ij}\ g_j >_0 \qquad f_i,\ g_j \in C_0^\infty(\Omega)$$

with

$$\underline{P} = \begin{bmatrix} 2L + a^2 & a \\ a & 2 \end{bmatrix} .$$

Since the coefficients of L are uniformaly bounded on Ω there exists a constant $k_1 > 0$ such that by Schwarz's Inequality:

$$|< f_1,\ L\ g_1 >_0| \leqq k_1 ||f_1||_p\ ||g_1||_p \qquad f_1,\ g_1 \in C_0^\infty\ (\Omega)$$

and

$$|\hat{V}(\underline{f},\underline{g})| \leqq 2\, k_1 ||f_1||_p\, ||g_1||_p + a^2 ||f_1||_0\, ||g_1||_0 + a\, ||f_1||_0\, ||g_2||_0 +$$

$$+\ a ||f_2||_0\, ||g_1||_0 + 2 ||f_2||_0\, ||g_2||_0$$

$$\leqq (2k_1 + a^2)\, ||f_1||_p\, ||g_1||_p + a ||f_1||_p\, ||g_2||_0 + a ||f_2||_0\, ||g_1||_p +$$

$$+\ 2\, ||f_2||_0\, ||g_2||_0$$

$$\leqq 2k_1\, ||f_1||_p\, ||g_1||_p + ||f_2||_0\, ||g_2||_0 + \{a ||f_1||_p +$$

$$+\ ||f_2||_0 \}\ \{ a ||g_1||_p + ||g_2||_0 \}$$

$$\leqq \{ \sqrt{2k_1} ||f_1||_p + ||f_2||_0 \}\{ \sqrt{2k_1}\, ||g_1||_p + ||g_2||_0 \} +$$

$$+\ \{ a ||f_1||_p + ||f_2||_0 \}\ \{ a ||g_1||_p + ||g_2||_0 \}$$

$$\leqq [\{\max(\sqrt{2k_1}, 1)\}^2 + \{\max(a,1)\}^2][||f_1||_p + ||f_2||_0][||g_1||_p +$$

$$||g_2||_0]$$

where we have used the fact that

$$||f_1||_p \geqq ||f_1||_0 \text{ and} ||g_1||_p \geqq ||g_1||_0 \quad .$$

Next let $k_2 = \{\max(\sqrt{2k_1}, 1)\}^2 + \{\max(a,1)\}^2$, and take the square on both sides of the inequality sign:

$$|\hat{V}(\underline{f},\underline{g})|^2 \leqq k_2^2\ \{||f_1||_p + ||f_2||_0\ \}^2 \{||g_1||_p + ||g_2||_0\ \}^2$$

$$= k_2^2\ \{||f_1||_p^2 + 2||f_1||_p\ ||f_2||_0 + ||f_2||_0^2\ \}\{||g_1||_p^2 + 2||g_1||_p\ ||g_2||_0 +$$

$$||g_2||_0^2\ \}$$

with the inequality:

$$2|a\ b| \leqq (\nu\ a^2 + \nu^{-1}\ b^2)$$

which is valid when $\nu > 0$, there follows

$$|\hat{V}(\underline{f},\underline{g})|^2 \leqq k_2^2\ \{(\nu + 1)||f_1||_p^2 + (1+\nu^{-1})||f_2||_0^2\}\{(\nu+1)||g_1||_p^2 +$$

$$+ (1+\nu^{-1})\ ||g_2||_0^2\}$$

$$\leqq k_2^2\ \{\max(\nu+1,\ 1+\nu^{-1})\}^2\ ||\underline{f}||_{p,0}^2\ ||\underline{g}||_{p,0}^2\ .$$

Thus for $D \geqq k_2\ \max(\nu+1,\ 1+\nu^{-1}) > 0$ there follows

$$|\hat{V}(\underline{f},\underline{g})| \leqq D||\underline{f}||_{p,0}\ ||\underline{g}||_{p,0}$$

where $||\cdot||_{p,0}$ denotes the norm induced by the inner product:

$$<\underline{f},\underline{g}>_{p,0} = \int_\Omega \{\sum_{|J|\leqq p} \partial^J f_1 \cdot \partial^J g_1 + f_2 \cdot g_2\} d\ x$$

for all $\underline{f},\underline{g}$ in $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$.

Furthermore $\hat{V}(\underline{f},\underline{f}) = 2 < f_1, Lf_1>_0 + a^2 < f_1,f_1>_0 +$

$$+ 2a< f_1,f_2>_0 + 2 < f_2,f_2>_0 \qquad \underline{f} \in C_0^\infty(\Omega) \times C_0^\infty(\Omega). \tag{VIII-19}$$

With Garding's Inequality (VII-9):

$$\hat{V}(\underline{f},\underline{f}) \geqq 2k < f_1,f_1>_p - 2K< f_1,f_1>_0 + a^2 < f_1,f_1>_0$$

$$+ 2a < f_1,f_2>_0 + 2 < f_2,f_2>_0 \qquad \underline{f} \in C_0^\infty(\Omega) \times C_0^\infty(\Omega).$$

And with (VIII-17) and (VIII-18)

$$\hat{V}(\underline{f},\underline{f}) \geqq 2\ \varepsilon < f_1,f_1>_p + a^2 < f_1,f_1>_0 + 2a < f_1,f_2>_0$$

$$+ 2 < f_2,f_2>_0 \qquad \underline{f} \in C_0^\infty(\Omega) \times C_0^\infty(\Omega).$$

Thus with $\varepsilon > 0$ and $a > 0$ there exists some constant $d > 0$ such that

$$\hat{V}(\underline{f},\underline{f}) \geqq d||\underline{f}||^2_{p,0} \qquad \text{for all } \underline{f} \in C_0^\infty(\Omega) \times C_0^\infty(\Omega).$$

By continuity $\hat{V}(\underline{f},\underline{g})$ can be extended to all of $H_0^p(\Omega) \times H_0^0(\Omega) = H(\Omega)$.

$$|V(\underline{f},\underline{g})| = |\hat{V}(\underline{f},\underline{g})| \leqq D||\underline{f}||_{p,0}||\underline{g}||_{p,0} \quad \underline{f},\underline{g} \in H(\Omega) \tag{VIII-20}$$

and $V(\underline{f},\underline{f}) \geqq d||\underline{f}||^2_{p,0} \qquad \underline{f} \in H(\Omega)$ (VIII-21)

Then by the Lax-Milgram Theorem there exists a symmetric, positive definite, bounded operator $\underline{S}$ $\varepsilon L(H,H)$ such that

$$V(\underline{f},\underline{g}) = <\underline{f},\underline{S}\ \underline{g}>_H = <\underline{S}\ \underline{f},\ \underline{g}>_H \qquad \underline{f},\ \underline{g} \in H(\Omega).$$

For functions $\underline{f},\underline{g} \in \mathcal{D}(\underline{W}) \subseteqq H$, since W is closed extension of L in H with domain $\mathcal{D}(W) = H_0^p(\Omega) \cap H^{2p}(\Omega)$ it follows from (VIII-19) that

$$V(\underline{f},\underline{g}) = <f_1,\ (2W + a^2)\ g_1>_0 + <f_1,\ a\ g_2>_0 +$$

$$+ <f_2,\ a\ g_1>_0 + <f_2,\ 2g_2>_0. \tag{VIII-22}$$

Let $v(\underline{f}) = V(\underline{f},\underline{f}) = <\underline{f},\underline{S}\ \underline{f}>_H$ for $\underline{f} \in H(\Omega)$ (VIII-23)

Next we evaluate $\dot{v}(\underline{f}) = 2V(\underline{W}\ \underline{f},\underline{f})$ for $\underline{f} \in \mathcal{D}(\underline{W})$. Notice that actually (VIII-23) must be taken for $V(\underline{f},\underline{f})$. $\dot{v}(\underline{f})$ can be taken as (VIII-22), if the proper restrictions are placed on W, so that $V(\underline{W}\ \underline{f},\underline{f}) = V(\underline{f},\ \underline{W}\ \underline{f})$. Taking first

$$V(\underline{W}\ \underline{f},\underline{f}) = 2<f_2,\ W f_1>_0 + a^2<f_2,f_1>_0 + a<f_2,f_2>_0 -$$

$$- a<W f_1,f_1>_0 - a^2<f_2,f_1>_0 - 2<W f_1,f_2>_0 - 2a<f_2,f_2>_0 =$$

$$= -a<W f_1,f_1>_0 - a<f_2,f_2>_0 \qquad \underline{f} \in \mathcal{D}(\underline{W}).$$

Next take

$$V(\underline{f}, \underline{\mathcal{W}}\,\underline{f}) = -a< f_1, W f_1>_0 + 2< f_1, W f_2>_0 + a^2< f_1, f_2>_0 -$$

$$- a^2< f_1, f_2>_0 - 2 < f_2, W f_1>_0 + a< f_2, f_2>_0 - 2a < f_2, f_2>_0 =$$

$$= -a< f_1, W f_1>_0 - a < f_2, f_2>_0 \qquad \underline{f} \in \mathcal{D}(\underline{\mathcal{W}})$$

if W is self-adjoint. Since

$$k_1< f_1, f_1>_p \geqq < W f_1, f_1>_0 \geqq \varepsilon < f_1, f_1>_p \qquad f_1 \varepsilon\, \mathcal{D}\,(W)$$

then, for any $a > 0$,

$$-e\, ||\underline{f}||^2_{p,0} \leqq \dot{v}(\underline{f}) = 2V(\underline{\mathcal{W}}\,\underline{f}, \underline{f}) \leqq -E||\underline{f}||^2_{p,0} \qquad \underline{f} \in \mathcal{D}(\underline{\mathcal{W}})$$

for some constants e and E, $e > E > 0$. Hence there exist some constants $\alpha = \frac{e}{d} > 0$ and $\beta = \frac{E}{D} > 0$, $\alpha > \beta > 0$ such that

$$- \alpha V(\underline{f}, \underline{f}) \leqq \dot{v}(\underline{f}) \leqq - \beta\, V(\underline{f}, \underline{f}) \qquad \underline{f} \in \mathcal{D}(\underline{\mathcal{W}}) \qquad \text{(VIII-24)}$$

In order to apply Theorem V-6, there remains to show that $\mathcal{R}(\lambda\, \underline{I} - \underline{\mathcal{W}}) = H^p_0(\Omega) \times H^0_0(\Omega)$ for real λ and $|\lambda|$ sufficiently large. Take any vector $\underline{g} = \mathrm{col}(g_1, g_2)$ in $H^p_0(\Omega) \times H^0_0(\Omega)$, then we must show that for $|\lambda|$ large there exists an $\underline{f} \in H^p_0(\Omega) \cap H^{2p}(\Omega) \times H^p_0(\Omega)$ such that $(\lambda\, \underline{I} - \underline{\mathcal{W}})\, \underline{f} = \underline{g}$. Thus let:

$$\lambda f_1 - f_2 = g_1 \in H^p_0(\Omega) \subseteqq H^0_0(\Omega)$$

$$W f_1 + (\lambda + a) f_2 = g_2 \in H^0_0(\Omega)$$

since $f_2 \in H^p_0(\Omega)$, $f_2 \varepsilon\, H^0_0(\Omega)$, thus substitution of f_2 in the second equation gives

$$W f_1 + \lambda\, (\lambda + a)\, f_1 = (\lambda + a)\, g_1 + g_2 = g \in H^0_0(\Omega).$$

For $|\lambda|$ sufficiently large $\lambda(\lambda + a)$ is always positive for $\lambda > 0$ and $\lambda < 0$. In Theorem VII-5, it has been proven that $\mathcal{R}(W) = H^0_0(\Omega)$. From the statements of the theorem it follows that $\mathcal{R}(W + \lambda(\lambda + a)I) = H^0_0(\Omega)$ for $|\lambda|$ sufficiently large. And thus $\mathcal{R}(\lambda\, \underline{I} - \underline{\mathcal{W}}) = H^p_0(\Omega) \times H^0_0(\Omega)$ for $|\lambda|$ sufficiently large. Thus by Theorem V-6 $\underline{\mathcal{W}}$ generates a group of exponential type $\{\underline{T}_t;\ t \in (-\infty,\infty)\}$ in $H_0^p(\Omega) \times H_0^0(\Omega)$ and there exist constants $\infty > M \geqq 1 \geqq m > 0$ such that

$$m\, e^{-\alpha t}||\underline{f}||_{p,0} \leqq ||\underline{T}_t\, \underline{f}||_{p,0} \leqq M\, e^{-\beta t}||\underline{f}||_{p,0}$$

And the conditions for asymptotic stability of the null solution $\underline{f} = \underline{0}$ of (VIII-16) are thus $a > 0$ and $k - \frac{K}{C_0} > \varepsilon > 0$ for some small $\varepsilon > 0$ where k and K are the constants from Garding's Inequality for τ, and C_0 is a constant given by the estimate

$$||f_1||_p^2 \geqq C_0||f_1||_0^2 \qquad\qquad f_1 \in \mathcal{D}(W).$$

This completes the proof of the theorem. Thus the main computational difficulty is find maximum values of k, C_0 and minimum values of K.

Next consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} - c\frac{\partial^2 u}{\partial x^2} + b\, u = 0 \qquad\qquad \text{(VIII-25)}$$

with a, b and c constants, $0 \leqq x \leqq 1$. And assume boundary conditions $u(t,0) = u(t,1) = 0$. The corresponding representation of (VIII-25) in terms of a closed operator W gives

$$\frac{d\,\underline{f}}{dt} = \underline{\mathcal{W}}\,\underline{f}$$

with

$$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f \\ f_t \end{bmatrix} \ ;\ \underline{\mathcal{W}} = \begin{bmatrix} 0 & 1 \\ -W & -a \end{bmatrix}$$

$\mathcal{D}(\underline{\mathcal{W}}) = H_0^1([0,1]) \cap H^2([0,1]) \times H_0^0\ ([0,1])$ and

$$W\, f_1 = -c\frac{\partial^2 f_1}{\partial x^2} + b\, f_1 \quad \text{for} \quad f_1 \in \mathcal{D}(W) = H_0^1([0,1]) \cap H^2([0,1]).$$

By Theorem VIII-1, the conditions for the asymptotic stability of the null solution $\underline{f} = \underline{0}$ are $a > 0$ and $< W f_1, f_1 >_0 \geqq \varepsilon < f_1, f_1 >_1$. Thus evaluating

$< W f_1, f_1 >_0 = \int_0^1 \{- c \frac{\partial^2 f_1}{\partial x^2} f_1 + b f_1^2\} d x$ on the $C_0^\infty([0,1])$ functions gives

$$< L f_1, f_1 >_0 = \int_0^1 \{ c (\frac{\partial^2 f_1}{\partial x})^2 + b f_1^2\} d x =$$

$$\geqq \varepsilon < f_1, f_1 >_1 + \{(c - \varepsilon)\pi^2 + b - \varepsilon\} < f_1, f_1 >_0$$

where the inequality

$$\int_0^1 (\frac{\partial f_1}{\partial x})^2 d x \geqq \pi^2 \int_0^1 f_1^2 d x$$

has been used. Thus the null solution to (VIII-25) is asymptotically stable for $a > 0$ and $\lim_{\varepsilon \to 0} \{(c \pi^2 + b) - \varepsilon(\pi^2 + 1)\} > 0$.

Hence for $c > 0$ and $b > - \pi^2 c$, $\underline{W}$ will generate a group $\{\underline{T}_t ; t \varepsilon(-\infty, \infty)\}$ of the exponential type in $H_0^1(\Omega) \times H_0^0(\Omega)$.

The conditions for asymptotic stability are thus identical to those found for the diffusion equation in Chapter VII. The solution to the wave equation possesses the group property, while the solution to the diffusion equation possesses only the semi-group property.

The eigenvalues λ for (VIII-3) were determined in Chapter VII as

$$\lambda_n = b + cn^2 \pi^2 \quad (n = \pm 1, \pm 2, \ldots).$$

Substitution in the first equation of (VIII-3) gives

$$\frac{d^2T}{dt^2} + a \frac{d T}{dt} + (b + cn^2 \pi^2) T=0 \quad n = \pm 1, \pm 2, \ldots$$

The eigenvalues $\mu_1(n)$ and $\mu_2(n)$ for the characteristic equation are thus

$$\mu_{1,2}(n) = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4(b + cn^2\pi^2)}.$$

Since the condition for asymptotic stability is Re $\mu(n) < 0$, it follows that a must satisfy $a > 0$. For $n \geq 1$, $b + cn^2\pi^2 > 0$ implies for all n, $c > 0$ and

$$b + c\,\pi^2 > 0.$$

Hence the result: $a > 0$, $c > 0$, $b > -c\,\pi^2$. This is the same as that just found by using Lyapunov stability theory.

IX. APPLICATIONS

In this chapter a number of applications are given. The emphasis in these applications is on the formulation of the problem in such a way that the formal mathematical operations yield rigorous mathematical results. The first example shows that the Lyapunov stability theory of Chapter V is applicable to systems of ordinary differential equations. In the next examples the emphasis is placed on the selection of the Lyapunov Functional, so that the application of well-known integral inequalities gives the maximal parameter range for stability of the null solution. Some results from Eckhaus[6] are demonstrated using the Lyapunov stability theory. The corresponding nonlinear cases are given in Chapter X.

Example IX-1. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2x_1 - 3x_2\end{aligned}\quad ; \ \underline{\dot{x}} = \underline{A}\,\underline{x} \ , \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \underline{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} . \qquad \text{(IX-1)}$$

$\underline{x} \in R^2$, $\underline{A} \in L(R^2, R^2)$

Obviously the solution $\underline{x} = \underline{0}$ is asymptotically stable, because the eigen - values of the coefficient matrix are -1 and -2.

Next introduce a norm:

$$||\underline{x}||_0^2 = \langle \underline{x},\underline{x} \rangle_0 = \underline{x}^T\underline{x} = x_1^2 + x_2^2 \ .$$

Then $\underline{A}$, a bounded linear operator with $\mathcal{D}(\underline{A}) = R^2$ and $\mathcal{R}(\underline{A}) = R^2$, is well defined on the Hilbert space $H_0 = (R^2; \langle \cdot,\cdot \rangle_0)$ and (IX-1) so defined belongs to the class (V-1).

Then take as positive definite bilinear functional $V(\underline{x},\underline{y}) = \langle \underline{x},\underline{y} \rangle_0$ for $\underline{x},\underline{y} \in R^2$. And the Lyapunov Functional $v(\underline{x})$ is:

$$v(\underline{x}) = V(\underline{x},\underline{x}) = \langle \underline{x},\underline{x} \rangle_0 \qquad \underline{x} \in R^2.$$

Thus

$$\frac{1}{2}\dot{v}(\underline{x}) = V(\underline{A}\ \underline{x},\underline{x}) = \langle \underline{A}\ \underline{x},\underline{x} \rangle_0 = -x_1x_2 - 3x_2^2$$

which is not even negative semi-definite for all $\underline{x} \in R^2$. Hence the inner product $\langle \cdot,\cdot \rangle_0$ does not provide us with a suitable Lyapunov Functional.

Next consider $V(\underline{x},\underline{y}) = \langle \underline{x},\underline{y} \rangle_1 = \langle \underline{x}, \underline{S}\ \underline{y} \rangle_0$, $\underline{x},\underline{y} \in R^2$,

with
$$\underline{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$

$\underline{S}$ is bounded, positive definite and symmetric, thus $\langle \cdot,\cdot \rangle_1$ is equivalent to $\langle \cdot,\cdot \rangle_0$. $v(\underline{x}) = \langle \underline{x},\underline{x} \rangle_1$, moreover

$$\frac{3-\sqrt{5}}{2} \langle \underline{x},\underline{x} \rangle_0 \leqq \langle \underline{x},\underline{x} \rangle_1 \leqq \frac{3+\sqrt{5}}{2} \langle \underline{x},\underline{x} \rangle_0. \tag{IX-2}$$

Then

$$\frac{1}{2}\dot{v}(\underline{x}) = \langle \underline{A}\ \underline{x}, \underline{x} \rangle_1 = -2x_1^2 - 3x_1x_2 - 2x_2^2$$

with

$$-\frac{7}{2} \langle \underline{x},\underline{x} \rangle_0 \leqq \langle \underline{A}\ \underline{x}, \underline{x} \rangle_1 \leqq -\frac{1}{2} \langle \underline{x},\underline{x} \rangle_0 . \tag{IX-3}$$

Combining (IX-3) with (IX-2) gives

$$-9.2\ V(\underline{x},\underline{x}) \leqq \dot{v}(\underline{x}) = 2V(\underline{A}\ \underline{x}, \underline{x}) \leqq -.191\ V(\underline{x},\underline{x}).$$

Since $\mathcal{R}(\underline{I} - \underline{A}) = R^2$, $\underline{A}$ generates a group of exponential type $\{\underline{T}_t\ ;\ t \in (-\infty, \infty)\}$ in $H_1 = (R^2, \langle \cdot,\cdot \rangle_1)$ such that

$$e^{-4.6t}||\underline{x}||_1 \leqq ||\underline{T}_t\underline{x}||_1 \leqq e^{-.0955t}\ ||\underline{x}||_1$$

And the origin $\underline{x} = \underline{0}$, is asymptotically stable. The selection of the bilinear functional $V(\underline{x},\underline{y})$ is crucial here.

The following two examples were studied by Eckhaus[6] using approximate methods. The linear cases are given in this chapter, the nonlinear cases are analyzed in Chapter X.

Example IX-2. Burgers' model to describe turbulence as studied by Eckhaus[6] is given by

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - u_1 - \frac{1}{R}\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_1^2}{\partial x} - u_1 u_2 = 0 \\ &\frac{\partial u_2}{\partial t} + \frac{1}{R} u_2 + \int_0^1 u_1^2 \, dx = 0 \end{aligned} \tag{IX-4}$$

$0 \leqq x \leqq 1$, R constant and boundary conditions $u_1(t,0) = u_1(t,1) = 0$.

The linearized system is:

$$\frac{\partial \underline{u}}{\partial t} = L\,\underline{u} \tag{IX-5}$$

where

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \; ; \quad L = \begin{bmatrix} 1 + \frac{1}{R}\frac{\partial^2}{\partial x^2} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}$$

$0 \leqq x \leqq 1$ and boundary conditions $u_1(t,0) = u_1(t,1) = 0$. The operator $\tau = -1 - \frac{1}{R}\frac{\partial^2}{\partial x^2}$ in L is a formally self-adjoint strongly elliptic partial differential operator for $R > 0$. In order to define L on a complete space we must introduce distributions. Notice that u_2 of (IX-4) is independent of x, thus L is defined in the $C_0^\infty([0,1]) \times R^1$ functions. $\underline{\Gamma}$, to be introduced next, will thus be defined on the Hilbert space $L^2([0,1]) \times (R^1; \langle\cdot,\cdot\rangle_0)$, where $\langle u,u\rangle_0 = u^2$. This space will formally be indicated by $L^2([0,1]) \; ; \underline{I})$.

Following Chapter VII, we get for (IX-5)

$$\frac{d\,\underline{f}}{d\,t} = -\,\underline{\mathcal{W}}\,\underline{f} \qquad \text{(IX-6)}$$

with

$$\underline{\mathcal{W}} = \begin{pmatrix} W & 0 \\ 0 & \frac{1}{R} \end{pmatrix} = \begin{pmatrix} -L & 0 \\ 0 & \frac{1}{R} \end{pmatrix}$$

for $\underline{f} \in \mathcal{D}(\underline{\mathcal{W}}) = H_0^1([0,1]) \cap H^2([0,1]) \times (R^1 ; \langle\cdot,\cdot\rangle_0)$. $\underline{\mathcal{W}}$ is a closed operator with $\mathcal{D}(\underline{\mathcal{W}}) \subsetneqq L^2([0,1] ; \underline{I})$ and $\mathcal{R}(\underline{\mathcal{W}}) = L^2([0,1] ; \underline{I})$. Hence (IX-6) belongs to the class (V-1).

Take as positive definite bilinear functional $V(\underline{f},\underline{g})$:

$$V(\underline{f},\underline{g}) = \langle \underline{f},\underline{g}\rangle_0 = \int_0^1 f_1 g_1\, dx + f_2 g_2 \quad \underline{f},\underline{g} \in L^2([0,1]; \underline{I}).$$

The Lyapunov Functional $v(\underline{f})$ becomes thus

$$v(\underline{f}) = V(\underline{f},\underline{f}) = \langle \underline{f},\underline{f}\rangle_0 \,. \qquad \underline{f} \in \mathcal{D}(\underline{\mathcal{W}})$$

The next step is to evaluate $\dot{v}(\underline{f}) = 2 \langle -\,\underline{\mathcal{W}}\,\underline{f},\underline{f}\rangle_0$ on the $C_0^\infty([0,1])$ functions:

$$\langle \mathcal{L}\underline{f},\underline{f}\rangle_0 = \int_0^1 \{f_1^2 + \frac{1}{R} f_1 \frac{\partial^2 f_1}{\partial x^2}\}d\,x - \frac{1}{R} f_2^2 =$$

$$= -\left[\int_0^1 \{ \frac{1}{R} (\frac{\partial f_1}{\partial x})^2 - f_1^2\}\, d\,x + \frac{1}{R} f_2^2\right] .$$

The integral inequality[31]

$$\int_0^1 (\frac{\partial f_1}{\partial x})^2\, d\,x \geqq \pi^2 \int_0^1 f_1^2\, d\,x \qquad \text{(IX-7)}$$

holds for the first term in the integrand, thus

$$\langle \mathcal{L}\,\underline{f},\underline{f}\rangle_0 \leqq -\left[\int_0^1 \{ \frac{\pi^2}{R} - 1)\, f_1^2\}d\,x + \frac{1}{R}\, f_2^2\right] .$$

Thus $-\,\underline{\mathcal{W}}$ is strictly dissipative for

$$0 < R < \pi^2$$

and $R(\underline{I} + \underline{W}) = L^2([0,1] \; ; \; \underline{I})$, hence by Theorem V-5 the null solution to (IX-6) is asymptotically stable. $\underline{W}$ generates a contraction semi-group $\{\underline{T}_t ; t \geqq 0\}$ in $L^2([0,1] \; ; \; \underline{I})$ with

$$||\underline{T}_t||_0 \leqq e^{-\alpha t} \qquad t \geqq 0,$$

$\alpha = \min \left(\frac{\pi^2}{R} - 1, \frac{1}{R} \right)$. This condition is identical to that obtained by requiring that λ_{min} of L be positive, since from Eckhaus[6]

$$\lambda_{1n} = \frac{\pi^2}{R} (n+1) - 1 \qquad n = 0,1,2, \ldots$$

$$\lambda_{20} = \frac{1}{R} .$$

Example IX-3. Next consider a second example from Eckhaus[6]:

$$\frac{\partial u}{\partial t} - \left(x^2 + \frac{2}{\sqrt{R}}\right) u - \frac{2}{\sqrt{R}} x \frac{\partial u}{\partial x} - \frac{1}{R} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \frac{\partial u^2}{\partial x} +$$

$$+ R^2 \left[\int_0^1 u^2 \, d x \right] u = 0 \qquad \text{(IX-8)}$$

$0 \leqq x \leqq 1$, R a positive constant, and boundary conditions $u(t,0) = u(t,1) = 0$. The linearized system is given by

$$\frac{\partial u}{\partial t} = L \, u = \left(x^2 + \frac{2}{\sqrt{R}}\right) u + \frac{2}{\sqrt{R}} x \frac{\partial u}{\partial x} + \frac{1}{R} \frac{\partial^2 u}{\partial x^2} \qquad \text{(IX-9)}$$

$0 \leqq x \leqq 1$ and boundary conditions $u(t,0) = u(t,1) = 0$. $\tau = -L$ is for $R > 0$ a strongly elliptic formal partial differential operator, which is, however, not formally self-adjoint. (IX-9) can again be formulated in terms of the operator differential equation (V-1) by letting:

$$\frac{d f}{dt} = - W f \qquad \text{(IX-10)}$$

with $W f = -L f$ for $f \in \mathcal{D}(W) = H_0^1([0,1]) \cap H^2([0,1])$. Take as positive definite bilinear functional $V(f,g)$:

$$V(f,g) = < f,g>_0. \qquad f,\ g \ \varepsilon\ L^2([0,1]).$$

Then we have to evaluate again $V(-W\ f,f)$ on the $C_0^\infty([0,1])$ - functions. Integration by parts and making use of the integral inequality (IX-7) gives

$$V(L\ f,f) \leqq - \left(\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - x^2 \right) < f,f>_0.$$

Thus a sufficient condition for the asymptotic stability of the null solution, f=0, is

$$\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - \max_{x\varepsilon[0,1]} \{x^2\} > 0$$

or

$$0 < R < \frac{1}{2} (1 + 2\pi^2 - \sqrt{1 + 4\pi^2}).$$

One can improve considerably on this last condition as an evaluation of the eigenvalues of λ suggest (see Eckhaus[(6)]) by observing that τ is equivalent to the strongly elliptic operator τ_e as given by:

$$\tau_e = - \frac{1}{w(x)} \frac{\partial}{\partial x} (p(x) \frac{\partial}{\partial x}) + q(x)$$

with

$$p = \exp \sqrt{R}\ \ x^2$$

$$w = R \exp \sqrt{R}\ \ x^2$$

$$q = - \frac{2}{\sqrt{R}} - x^2.$$

An inner product $<\cdot,\cdot>_w$ which is equivalent to $<\cdot,\cdot>_0$ can now be chosen for the positive definite bilinear functional $V(f,g)$:

$$V(f,g) = < f,w(x)\ g>_0 = < f,g>_w \qquad f,g\ \varepsilon\ L^2([0,1]).$$

It can easily be checked that W_e, corresponding to τ_e, is self-adjoint with respect to $<\cdot,\cdot>_w$. Evaluation of $V(-W_e f,f) = <-W_e f,f>_w$ on the $C_0^\infty([0,1])$ - functions gives

$$V(-\tau_e f,f) = -\int_0^1 \{e^{\sqrt{R}\,x^2}\left(\frac{\partial f}{\partial x}\right)^2 - R\, e^{\sqrt{R}\,x^2}\left(x^2+\frac{2}{\sqrt{R}}\right)f^2\}d\,x.$$

The integral inequality (IX-7) can now be applied to $e^{\frac{1}{2}\sqrt{R}\,x^2} f$, rather than to f. Substitution gives:

$$V(-\tau_e f,f) \leqq -\left(\frac{\pi^2}{R}-\frac{1}{\sqrt{R}}\right) < f,f>_w .$$

The sufficient condition for the asymptotic stability of the null solution to (IX-10) becomes then

$$0 < R < \pi^4.$$

As should be expected, this condition is identical to that found by evaluating the eigenvalues of τ. The choice of bilinear functional $V(f,g)$ is thus seen to be very important for the determination of a maximal parameter range for asymptotic stability. The equivalence of $<\cdot,\cdot>_0$ and $<\cdot,\cdot>_w$ indicates that the stability properties of (IX-10) are the same for both. The selection of the bilinear functional is also important, when dealing with nonlinear systems. This will be shown in Chapter X.

The following example shows the effect of the elimination of the highest order odd derivative of τ on the parameter range for which the system is stable, by introducing a τ_e.

Example IX-4. Consider the system

$$\frac{\partial u}{\partial t} = -\left(\frac{1}{R^2}\frac{\partial^4 u}{\partial x^4}+\frac{1}{R\sqrt{R}}\frac{\partial^3 u}{\partial x^3}+\frac{5}{4R}\frac{\partial^2 u}{\partial x^2}+\frac{1}{\sqrt{R}}\frac{\partial u}{\partial x}+\frac{1}{4}u\right) = L\,u. \qquad \text{(IX-11)}$$

$0 \leqq x \leqq 1$, $R > 0$ and boundary conditions

$$u(t,0) = u(t,1) = \left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0.$$

$\tau = -L$ is a strongly elliptic formal partial differential operator. (IX-11) can be formaulated as (V-1) by defining

$$\frac{d\,f}{dt} = -\,W\,f \tag{IX-12}$$

with $W\,f = -L\,f$ for $f \in \mathcal{D}\,(W) = H_0^2([0,1]) \cap H^4([0,1])$. As $V(f,g)$ can be taken:

$$V(f,g) = \langle f,g\rangle_0 \qquad f,g \in L^2([0,1]).$$

Then an evaluation of $V(-W\,f,f)$ on the $C_0^\infty([0,1])$ -functions results in a sufficient condition for asymptotic stability of the solution f=0 of (IX-12):

$$0 < R < \frac{4}{5}\,\pi^2.$$

However, τ is equivalent to

$$\tau_e = +\,\frac{1}{w(x)}\;\frac{\partial^2}{\partial x^2}\;(p(x)\,\frac{\partial^2}{\partial x^2}) + \frac{1}{R}\;\frac{\partial^2}{\partial x^2} + \frac{1}{\sqrt{R}}\,\frac{\partial}{\partial x} + \frac{1}{4}$$

with $p(x) = e^{\frac{1}{2}\sqrt{R}\;x}$

$$w(x) = R^2\;e^{\frac{1}{2}\sqrt{R}\;x}$$

τ_e is also strongly elliptic. The bilinear functional $V(f,g)$ should now be taken as:

$$V(f,g) = \langle\, f,w(x)g\,\rangle_0 = \langle\, f,g\rangle_w \qquad f,g \in L^2([0,1])$$

$\langle\cdot,\cdot\rangle_0$ and $\langle\cdot,\cdot\rangle_w$ are again equivalent. Evaluation of $V(-W_e f,f) = \langle -W_e f,f\rangle_w$ shows that the sufficient condition for asymptotic stability of the null solution to (IX-12) can be extended to

$$0 < R < \frac{16}{15}\,\pi^2.$$

The integral inequality (IX-7) is here applied to $e^{\frac{1}{4}\sqrt{R}\;x}\;\frac{\partial f}{\partial x}$ and $e^{\frac{1}{4}\sqrt{R}\;x}\;f$ rather than to $\frac{\partial f}{\partial x}$ and f respectively.

The above examples illustrate clearly the application of the stability theory developed here. The importance of the equivalent inner products is demonstrated for the determination of maximal parameter ranges for asymptotic stability of the null solutions. In each case the stability problem has been formulated in such a way that all mathematical operations can be formally carried out and these formal operations are rigorously justified.

X. SOME NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

The important advantage of Lyapunov's Direct Method over approximate methods in the stability analysis of finite dimensional systems of nonlinear ordinary differential equations is that nonlinearities can be introduced easily and without lack of mathematical rigor. In applications, the main emphasis is on the so-called "Lur'e type" nonlinearities, frequently encountered in control system applications. Similar nonlinearities are found in such distributed parameter systems as nuclear reactor systems and quantum physics.

Here again the main problem is that of the existence and uniqueness of the solutions, thus a problem in the theory of partial differential equations itself. In the following examples, the existence of solutions is not rigorously established; instead the following assumptions are made:

1. The solutions to the linearized system exist and belong to a Hilbert space H.

2. The linearized system is asymptotically stable.

3. The solutions to the nonlinear system exist and belong for all $t \geqq 0$ to H.

The sufficient conditions for stability can then be based on estimates for the nonlinear terms in the derivative of the Lyapunov Functional.

<u>Example X-1</u>. Consider first the non-linear case of Example IX-2 and suppress (as in Eckhaus[6]) the $\frac{\partial u_2}{\partial t}$ term of the second equation of (IX-4). This gives:

$$\frac{\partial u}{\partial t} = u + \frac{1}{R}\frac{\partial^2 u}{\partial x^2} - \frac{\partial u^2}{\partial x} - R\left[\int_0^1 u^2 \, d x\right] u = L u \qquad \text{(X-1)}$$

$0 \leqq x \leqq 1$ and boundary conditions $u(t,0) = u(t,1) = 0$. And let R be a positive constant. The linearized system is asymptotically stable for

$$0 < R < \pi^2.$$

Define (X-1) again on a complete space, which is assumed to be the space for the linearized system, then

$$\frac{d\,f}{dt} = -\,W\,f$$

with $W\,f = -\,L\,f$ for f sufficiently smooth. The domain of the linearized differential operator is taken as $\mathcal{D}(W) = H_0^1([0,1]) \cap H^2([0,1])$.

Take as bilinear functional $V(f,g)$:

$$V(f,g) = \langle f,g\rangle_0 \qquad f,g \in L^2([0,1]).$$

Then there remains to evaluate $\dot{v}(f) = 2\langle L\,f,f\rangle_0$ on the $C_0^\infty([0,1])$ functions, thus

$$\langle L\,f,f\rangle_0 = -\int_0^1 \{-f^2 - \frac{1}{R}\,f\,\frac{\partial^2 f}{\partial x^2} + f\,\frac{\partial f^2}{\partial x} + R(\int_0^1 f^2\,d\,x)f^2\}d\,x. \quad f \in \mathcal{D}(W).$$

Integration by parts and substitution of the boundary conditions gives:

$$\langle L\,f,f\rangle_0 = -\int_0^1 \{-\,f^2 + \frac{1}{R}\,(\frac{\partial f}{\partial x})^2 + R\,(\int_0^1 f^2\,d\,x)f^2\}\,d\,x \qquad f \in \mathcal{D}(W).$$

Application of the integral inequality (IX-7), the fact that $\int_0^1 f^2\,dx \geqq 0$ for $f \in \mathcal{D}(W)$ and $R > 0$ gives:

$$\langle L\,f,f\rangle_0 \leqq -\,(\frac{\pi^2}{R} - 1)\,\langle f,f\rangle_0 \qquad f \in \mathcal{D}(W).$$

Thus the modified nonlinear system has, under the stated assumptions, an asymptotically stable null solution for

$$0 < R < \pi^2.$$

This verifies the results obtained by Eckhaus[6].

In [33] it is shown that a similar result is obtained if the $\frac{\partial u_2}{\partial t}$ is not suppressed.

Example X-2. Next consider the nonlinear case of Example IX-3. The system is given by

$$\frac{\partial u}{\partial t} = (x^2 + \frac{2}{\sqrt{R}})\, u + \frac{2}{\sqrt{R}}\, x \frac{\partial u}{\partial x} + \frac{1}{R} \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \frac{\partial u^2}{\partial x} - R^2 \left[\int_0^1 u^2\, d\, x\right] u = L\, u \qquad \text{(X-2)}$$

$0 \leqq x \leqq 1$, boundary conditions $u(t,0) = u(t,1) = 0$ and R a positive constant. The linearized system is asymptotically stable for $0 < R < \pi^4$.

The next step is to introduce a formulation of (X-2) on a complete space:

$$\frac{d\, f}{dt} = -\, W\, f \qquad \text{(X-3)}$$

with $W\, f = -\, L\, f$ for f sufficiently smooth. The domain of the linearized differential operator is taken as $\mathcal{D}(W) = H_0^1([0,1]) \cap H^2([0,1])$.

The bilinear functional V(f,g) will be taken as

$$V(f,g) = < f, w(x)\; g>_0 = < f,g>_w \qquad f,g \in L^2([0,1])$$

with $w(x) = R \exp \sqrt{R}\, x^2$. For the linear part of W we must substitute the equivalent differential operator

$$\tau_e = -\, \frac{1}{w(x)}\, \frac{\partial}{\partial x} (p(x)\, \frac{\partial}{\partial x}) + q(x)$$

where

$$p = \exp \sqrt{R}\, x^2$$

$$q = \frac{-2}{\sqrt{R}} - x^2.$$

Evaluation of $V(L.f,f)$ on the $C_0^\infty([0,1])$-functions gives

$$V(L\; f,f) = -\int_0^1 \{ e^{\sqrt{R}\, x^2} (\frac{\partial f}{\partial x})^2 - R\, e^{\sqrt{R}\, x^2} (x^2 + \frac{2}{\sqrt{R}})\, f^2 +$$

$$+ \frac{1}{2} R\, e^{\sqrt{R}\, x^2} f \frac{\partial f^2}{\partial x} + R^3\, e^{\sqrt{R}\, x^2} \left[\int_0^1 f^2\, d\, x\right] f^2 \;\; d\, x.$$

Integration by parts, substitution of the boundary conditions and the inequality (IX-7) together with the fact that $\int_0^1 f^2\, d\, x \geqq 0$ and $R > 0$ give

$$V(L\; f,f) \leqq -\int_0^1 \left[\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - \frac{2}{3}\sqrt{R}\;\; xf\right] f^2\, w(x)\, d\, x.$$

Since $0 \leqq x \leqq 1$, L will certainly be strictly dissipative with respect to $\langle \cdot , \cdot \rangle_w$ for

$$\frac{\pi^2}{R} - \frac{1}{\sqrt{R}} - \frac{2}{3} \; R \, |f| > 0$$

for all $x \, \varepsilon [0,1]$. Thus the null solution of (X-3) will be asymptotically stable for all initial distrubances bounded by

$$\max_{x \varepsilon [0,1]} |f| < \frac{3}{2} \, \frac{1}{R} \left(\frac{\pi^2}{\sqrt{R}} - 1 \right) \quad \text{for all } f \text{ sufficiently smooth.}$$

This result is again similar to that obtained by Eckhaus[6], however, the use of Lyapunov stability theory enables one to obtain it in a straightforward way, without making many complicated calculations as is the case when using asymptotic expansions.

The assumptions made concerning the existence of solutions are not more stringent than those made as justification for the use of approximate methods. However, the necessity of a self-contained Lyapunov stability theory for certain classes of nonlinear partial differential equations remains. To what extent the theory of monotone operators[22] enables an extension of the stability theory, developed here for dissipative operators, to certain classes of nonlinear operators is left as a suggestion for further research.

XI. CONCLUSIONS AND SUGGESTED FURTHER RESEARCH

A. Conclusions

The objective of this research is to establish a Lyapunov stability theory for solutions to partial differential equations on a mathematically rigorous basis. Because of the complex mathematical nature of partial differential equations, a type of synthesis method has been developed.

Lyapunov's Direct Method for ordinary differential equations is generalized to a class of operator differential equations. This generalization is based on the fundamental solution structure of groups and semi-groups, which is not restricted to ordinary differential equations.

Once the stability properties are formulated in terms of the group and semi-group structure, it becomes possible to impose the conditions for their stability on the corresponding infinitesimal generators. This enables the formulation of a Lyapunov stability theory, analogous to Lyapunov's Direct Method for ordinary differential equations, for a large class of operator differential equations--specifically, for those bounded and unbounded operators which are the infinitesimal generators of contraction groups and semi-groups. The conditions for stability and asymptotic stability for this class of operators is based on the Hilbert space theory of dissipative operators.

The Hilbert space structure, associated with dissipative operators led to the pivotal notion of equivalent inner products, which enables one to relate dissipativity in a satisfactory way to stability, a norm property. Sufficient conditions are established for the stability and asymptotic stability of semi-groups and for the null solution of the corresponding operator differential equations. For groups this is extended to necessary and sufficient conditions for stability and asymptotic stability, respectively.

The next development is that of defining partial differential equations in terms of the general operator differential equation. For this purpose a formal linear partial differential operator is introduced. The formal partial differential operator is extended to an operator defined in a complete space by introducing distributions. This is followed by associating the partial differential equation with a boundary value problem. In order to maintain the general operator differential equation structure this is only carried out for the class of strongly elliptic partial differential operators satisfying the Dirichlet boundary conditions. Subsequently, a formulation of a class of evolution equations and a class of wave equations is obtained in terms of the general operator differential equation.

These two classes of equations are very important in physics and engineering applications, and the formulation as operator differential equations enables us to apply the developed Lyapunov stability theory for a rigorous determination of their stability properties.

The relation between the stability theory and the group and semigroup structure automatically gives the existence and uniqueness properties for the solutions of these equations. For the evolution equation a general Lyapunov Functional is formulated in terms of the inner product of the Hilbert space on which the operator is defined. Subsequent applications clearly reveal the importance of the notion of equivalent inner products and its consequences for obtaining maximal parameter ranges for stability.

For a class of wave equations a Lyapunov Functional is developed, which not only clearly gives conditions for stability and asymptotic stability of the null solution, but also exhibits the group structure of the solutions.

In a final chapter the stability properties of some nonlinear systems are formally investigated. Except for these last results, the emphasis is on a mathematically rigorous approach to the important problem of stability of solutions to partial differential equations. That such an approach gives only results for a limited class of partial differential equations is not surprising. The theory of partial differential equations itself is a field of extensive research in mathematics and it is forseen that many of its developments might find applications in stability studies. Some suggestions for further research are given in the next section.

B. Suggested Further Research

Future research in the stability properties of solutions to partial differential equations can progress in many directions. Along the lines of the stability theory developed in this research it must be pointed out that only a small class of linear time-invariant partial differential equations have been formulated in terms of the general operator differential equation. The possibilities of extending this class must be explored, even though the rigid structure might have its limitations.

There is also a need for investigating the possibilities of extending the developed stability theory to those linear time-varying operator differential equations, which might possibly generate two parameter groups and semi-groups.

The potential use of Lyapunov stability theory to establish rigorously the stability conditions of nonlinear systems justifies a continued effort to extend the results for linear systems. The natural extension of linear disspative operators to the nonlinear case seem to be the monotone operators[22] The suggestion is that there might exist a nautral extension of the stability theory for linear dissipative operators to one for monotone operators.

The stability theory developed is based on the Hilbert space structure. The stability results are all with respect to norms induced by an inner product. This provides no limitation for finite dimensional systems, since all norms are equivalant. The question must then be raised as to how far this equivalence of norm principle can be carried through for infinite dimensional systems. In other words, which other norms are equivalent to the one induced by an inner product? Is there a Banach space structure other than the Hilbert spaces that encompasses more equivalent norms and hence a broader stability analysis?

The stability properties obtained are generally those with respect to the L^2-norm. The extent to which this kind of stability implies stability in the classical sense, i.e., with respect to a C^p-norm should be investigated. This is the direction indicated by Conjecture VII-1. Sobolev's Imbedding Theorems will play an important role in such an evaluation.

Even though partial differential equations appear in many engineering applications it is plain that the suggested research problems are very mathematically orientated. Unfortunately, the complexity of partial differential equations seems to require a highly specialized mathematical background in functional analysis, topology, generalized functions, etc. It is hoped that this will not deter others from investigating the very important problem of the stability of solutions to partial differential equations in the future.

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PART II

CONTRACTION GROUPS AND EQUIVALENT NORMS

by

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ABSTRACT

In this report, necessary and sufficient conditions are obtained for a closed linear operator A to generate a group $\{T_t; -\infty < t < \infty\}$ in a Banach space such that $\{T_t; t \geq 0\}$ is a negative contractive semi-group with respect to an equivalent norm. These results are refined to the case of a group in Hilbert space.

It is well known that some systems of differential equations, both ordinary and partial, can be reduced to the form

$$\frac{dx}{dt} = Ax \qquad (x\varepsilon\mathcal{D}(A)) \tag{1}$$

where A is a linear operator with domain, $\mathcal{D}(A)$, and range, $\mathcal{R}(A)$, both in a real B-space, X. If A is the infinitesimal generator of a semi-group $\{T_t;\ t \geqq 0\} \subseteqq L(X,X)$ of class (C_0), a solution to (1) starting at $t = 0$ from $x = x_0\varepsilon\mathcal{D}(A)$ is given by $x(t;\ x_0) = T_t x_0$ for $t \geqq 0$ with $x(0;\ x_0) = x_0$ (1). If A is the infinitesimal generator of a group, then the above solution is valid for $-\infty < t < \infty$.

Criteria in terms of the operator A (in other words, properties of the coefficients of the original system of differential equations) which would enable one to deduce the existence as well as asymptotic behavior of solutions are desirable. One such result already exists for A to be the infinitesimal generator of a semi-group (Theorem 1). The object of this report is to refine these results to groups of certain types. The development is in terms of real B-spaces but can easily be extended to complex spaces.

DEFINITION 1

Let $\{T_t\}$ be a semi-group of bounded linear operators in a B-space. If $||T_t|| \leqq N$ then the semi-group is said to be equibounded; if $N = 1$ then $\{T_t\}$ is called a contraction semi-group. If there exist finite $M > 0$ and $\beta > 0$ such that $||T_t|| \leqq Me^{-\beta t}$ then $\{T_t\}$ is said to be a negative semi-group; if $M = 1$ the semi-group is called negative contractive.

An important property of B-spaces is that a semi-scalar product [.,.] can be defined on any B-space such that $[x,y] = f_y(x)$ where f_y is a linear continuous functional such that $f_y(y) = ||y||^2$ (2)

DEFINITION 2

Let X be a B-space with norm $||\cdot||$ and let $[\cdot,\cdot]$ be a semi-scalar product on X. Then the semi-scalar product $[\cdot,\cdot]_1$ is said to be equivalent to $[\cdot,\cdot]$ on X iff $||\cdot||_1$ and $||\cdot||$ are equivalent norms on X.

The class of operators A needed for the statement of our results is described in the following definition[1].

DEFINITION 3

Let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ contained in a real B-space X. A is called dissipative with respect to the semi-scalar product $[\cdot,\cdot]$ if $[Ax, x] \leqq 0$ whenever $x\varepsilon\mathcal{D}(A)$ and strictly dissipative if there exists a $\gamma > 0$ such that

$$[Ax, x] \leqq -\gamma\, [x, x] = -\gamma||x||^2 \quad (x\varepsilon\mathcal{D}(A)) \tag{2}$$

The following results are due to Lumer and Phillips [2].

THEOREM 1

Let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ contained in a real B-space X and such that $\mathcal{D}(A)$ is dense in X. Then A generates a contraction semi-group in X iff A is dissipative (with respect to any semi-scalar product) and $\mathcal{R}(I - A) = X$.

COROLLARY

If A is a densely defined closed linear operator such that $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are contained in a B-space X and if A and its dual operator A' are both dissipative, then A generates a contraction semi-group.

In the course of the proof of Theorem 1 it is deduced that the resolvent operator of A, $R(\lambda; A)$, exists for all $\lambda > 0$ and satisfies the estimate $||R(\lambda; A)|| \leqq \lambda^{-1}$. From this fact we can deduce a useful criterion for $\{T_t\}$

to be a group and the solution $T_t x_0$ of (1) to approach the null solution as $t \to \infty$. We begin by considering semi-groups.

THEOREM 1'

Let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ both in a real B-space X such that $\mathcal{D}(A)$ is dense in X. Then A generates a negative contractive semi-group in X iff A is strictly dissipative and $\mathcal{R}(I(1-\gamma) - A) = X$ where γ is the constant appearing in (2).

PROOF

Apply Theorem 1 to the dissipative operator $B = A + \gamma I$. The result follows upon noting that B generates $\{S_t;\ t \geqslant 0\}$ iff $S_t = e^{\gamma t} T_t$ where $\{T_t\}$ is generated by A.

From the remark preceding Theorem 1' it follows that $R(\mu;\ A)$ exists for all $\mu > -\gamma$ and $||R(\mu;A)|| \leqq (\mu + \gamma)^{-1}$. The corollary of Theorem 1 can also be extended to strictly dissipative operators.

LEMMA 1

Let A be the infinitesimal generator of an equibounded (negative) semi-group $\{T_t;\ t \geqq 0\}$ in a real B-space $(X, ||\cdot||_1)$. Then there exists an equivalent semi-scalar product, $[\cdot,\cdot]$ inducing an equivalent norm $||\cdot||_2$ with respect to which A is (strictly) dissipative.

PROOF

By hypothesis $||T_t||_1 \leqq Me^{-\alpha t}$ where $\alpha \geqq 0$ is a constant and the constant M may be taken to be $M > 1$. If we set $S_t = e^{\alpha t} T_t$ then $\{S_t;\ t \geqq 0\}$ is a semi-group satisfying $||S_t||_1 \leqq M$. Let $x \in X$ and define $||\cdot||_2$ by

(3) $$||x||_2 = \sup_{t \geqq 0} ||S_t x||_1$$

In [3], it is shown that

(4) $$||x||_1 \leqq ||x||_2 \leqq M||x||_1 \qquad (x\varepsilon X)$$

(5) $$||S_t||_2 \leqq 1 .$$

By the remark following Definition 1, there exists a semi-scalar product $[.,.]$ consistent with $||\cdot||_2$ such that, by Theorems 1 and 1',

$$[(A + \alpha I)x, x] \leqq 0 \qquad (x\varepsilon \mathcal{D}(A)).$$

This proves Lemma 1 and establishes a starting point for groups in B-spaces, and as a special case, Hilbert spaces.

THEOREM 2

Let A be a linear operator with $\mathcal{D}(A)$ and $\mathcal{R}(A)$ both contained in a real B-space $(X, ||\cdot||_1)$ such that $\mathcal{D}(A)$ is dense in X. Then A generates a group $\{T_t; -\infty < t < \infty\}$ in X such that $\{T_t; t \geqq 0\}$ is a negative contractive semi-group with respect to an equivalent norm $||\cdot||$ iff

(6) $$-\delta\, ||x||^2 \leqq [Ax, x] \leqq -\gamma||x||^2 \qquad (x\varepsilon\mathcal{D}(A))$$

where $\infty > \delta \geqq \gamma > 0$ and $[.,.]$ is an equivalent semi-scalar product consistent with $||\cdot||$, and

(7) $$\mathcal{R}(I(1-\gamma) - A) = X, \qquad \mathcal{R}(I(1+\delta) + A) = X.$$

PROOF

Suppose that (6) and (7) are valid. Then $B = A + \gamma I$ and $C = -A - \delta I$ are dissipative. As in the proof of Theorem 1' it follows that $\{T_t; t \geqq 0\}$ is a negative contractive semi-group. Moreover $R(\mu; A)$ exists for all $\mu > -\gamma$ and $||R(\mu; A)|| \leqq (\mu + \gamma)^{-1}$. Similarly $R(\mu; A)$ exists for all $\mu < -\delta$ and $||R(\mu; A)|| \leqq (|\mu| - \delta)^{-1}$. These last two conditions imply that A is also the infinitesimal generator of a group[1].

Conversely suppose that A generates a group such that $\{T_t;\ t \geq 0\}$ is a negative contractive semi-group with respect to $||\cdot||$, that is, $||T_t|| \leq e^{-\beta t}$ $(t \geq 0)$ where $\beta > 0$. It is known that for a group $||T_t^{-1}|| \leq Me^{\alpha t}$, where $M \geq 1$ and α can be chosen such that $\alpha \geq \beta$ (1). Define $S_t = T_t^{-1}e^{-\alpha t}$ and define $||\cdot||_2$ as in (3) but using $||\cdot||$ on the right of (3). Then $||S_t||_2 \leq 1$ and $||\cdot||_2$ is equivalent to $||\cdot||$ and so equivalent to the original norm. It follows from (5) that $[S_t x - x, x]_2 \leq 0$ and so upon dividing by t and letting $t \downarrow 0$ we have

$$[(-A - \alpha I)x,\ x]_2 \leq 0.$$

This yields an equivalent semi-scalar product and the left side of inequality (6) with $\delta = \alpha$. To show that the right side is also valid consider

$$(8) \qquad [T_s e^{\beta s} x - x, x] \leq ||T_s e^{\beta s} x||_2\ ||x||_2 - ||x||_2^2 .$$

Next estimate $||T_s e^{\beta s} x||_2$ as follows

$$||T_s e^{\beta s} x||_2 = \sup_{t \geq 0} \{ e^{(\beta-\alpha)s} ||T_{s-t} e^{\alpha(s-t)} x|| \} =$$

$$= e^{(\beta-\alpha)s} \max(\sup_{0 \leq t \leq s} ||T_t e^{\alpha t} x||,\ ||x||_2).$$

Since

$$\sup_{0 \leq \tau \leq s} ||T_\tau e^{\alpha\tau} x|| \leq \sup_{0 \leq \tau \leq s} e^{(\alpha-\beta)\tau} ||x|| \leq e^{(\alpha-\beta)s} ||x||$$

it follows that

$$||T_s e^{\beta s} x||_2 \leq \max(||x||,\ e^{(\beta-\alpha)s} ||x||_2) \leq ||x||_2 .$$

Hence (8) yields

$$[T_s e^{\beta s}\ x - x, x]_2 \leq 0$$

which in turn implies the right side of (6) with $\gamma = \beta$.

Finally (7) follows from Theorem 1 applied to the dissipative operators $T_{-t}e^{\alpha t}$(with respect to $||\cdot||_2$) and $T_t e^{\beta t}$ (with respect to $||\cdot||_1$)

COROLLARY

If A is a densely defined closed linear operator such that $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are both contained in a B-space X and if A and its dual operator A' satisfy (6), then A generates a group such that $\{T_t;\ t \geqq 0\}$ is a negative contractive semi-group.

Theorem 2 can be strengthened in Hilbert spaces so that it holds for scalar products. The proof is not a straightforward application of Theorem 2. The difficulty lies in the fact that if (H, (.,.)) is a Hilbert space with scalar product (.,.) then H with an equivlanet norm is not necessarily a Hilbert space. For example, a Euclidean 2-space $(X,\ ||\cdot||)$ with $||(x_1,x_2)||^2 = x_1^2 + x_2^2$ is a Hilbert space while $(X,\ ||\cdot||_1)$, where $||(x_1,\ x_2)||_1 = |x_1| + |x_2|$, is not a Hilbert space.

We require the following lemma.

LEMMA 2

Let (H, (.,.)) be a real Hilbert space and $(T_t;\ t \geqq 0)$ a semi-group on H with infinitesimal generator A; then

$$\lim_{t\to 0+} t^{-1}((T_t x,\ T_t x) - (x,x)) = 2(Ax,x) \quad (x \varepsilon D(A)).$$

PROOF

The result follows from the identity

$$(T_t x,\ T_t x) - (x,x) = (T_t x,\ T_t x) - (x,\ T_t x) + (x,\ T_t x) - (x,x)$$

and the continuity of the scalar product.

THEOREM 3

Theorem 2 is valid for (H, (.,.)) a real Hilbert space and [.,.] an equivalent scalar product with $[x,x] = ||x||^2$ and $(x,x) = ||x||_1^2$.

PROOF

It is only necessary to prove that if A generates a group such that $\{T_t; t \geqq 0\}$ is a negative semi-group with respect to $||\cdot||_1$, then an equation of the form (6) is valid where $[.,.]$ is an equivalent scalar product. Define $[.,.]$ by

$$[x, y] = \int_0^\infty (T_t x, T_t y)\, dt \tag{9}$$

By hypothesis, $||T_t||_1 \leqq Me^{-\beta t}$ $(t \geqq 0)$, where $\beta > 0$ and $M \geqq 1$; hence

$$[x,x] = \int_0^\infty ||T_t x||_1^2\, dt \leqq \int_0^\infty M^2 e^{-2\beta t} ||x||_1^2\, dt = (M^2/2\beta)||x||_1^2 \tag{10}$$

Since $\{T_t\}$ is a group, there exist constants $\alpha \geqq \beta$ and $1/k \geqq 1$ such that $||T_t^{-1}||_1 \leqq (1/k)e^{\alpha t}$. By using the fact that $||T_t x||_1 \geqq ||T_t^{-1}||_1^{-1}\ ||x||_1$ it follows from (9) that

$$[x,x] \geqq \int_0^\infty k^2 e^{-2\alpha t} ||x||_1^2\, dt = (k^2/2\alpha)||x||_1^2 \tag{11}$$

We leave it to the reader to verify that $[.,.]$ is a scalar product. The equivalence of the two scalar products follows from (10) and (11).

To show that an equation of the form (6) is valid we consider

$$[T_t\, x,\ T_t x] - [x,\ x] = \lim_{n\to\infty}\Big[\int_0^n (T_s T_t x,\ T_s T_t x)ds - \int_0^n (T_s x, T_s x)ds\Big]$$

$$= \lim_{n\to\infty}\Big[\int_n^{n+t} (T_s x,\ T_s x)\ ds\Big] - \int_0^t (T_s x,\ T_s x)ds$$

$$= -\int_0^t (T_s x,\ T_s x)ds, \qquad (t > 0).$$

This last equality and Lemma 1 imply that

$$2[Ax,\ x] = -\ ||x||_1^2 \qquad (x \in \mathcal{D}(A)) \tag{12}$$

Equations (10), (11), and (12) yield (6) with $\gamma = \beta/M^2$ and $\delta = \alpha/k^2$.

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